The University of Sydney
School of Mathematics and Statistics

Tutorial 13 (Week 13)

MATH1014: Introduction to Linear Algebra — Semester 2, 2009

Revision

Vectors

1. Let $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix}$.

Find

(a) $2\mathbf{u} - 5\mathbf{v}$; (b) $\mathbf{u} \cdot \mathbf{u}$; (c) $\mathbf{u} \cdot \mathbf{v}$; (d) $||\mathbf{u}||$; (e) $||\mathbf{v}||$; (f) $\mathbf{u} \times \mathbf{v}$;
(g) a unit vector in the direction of $\mathbf{u}$; (h) the angle between $\mathbf{u}$ and $\mathbf{v}$.

2. Let $P = (2, 1, 0)$, $Q = (-3, 4, 2)$ and $R = (1, 2, -5)$ be points in space.

(a) Find a vector equation, and parametric equations, for the straight line through P and Q.

(b) Find a normal to the plane containing the three points P, Q and R.

(c) Find a Cartesian equation for the plane containing the three points P, Q and R.

Modular arithmetic

3. Calculate

(a) $1 + 2 + 3 + 4 + 5 + 6 + 7$ in $\mathbb{Z}_8$;

(b) $(5 + 9) \times (8 + 6)$ in $\mathbb{Z}_{10}$;

(c) 2^{501} in $\mathbb{Z}_5$.

4. Find the check digit, d, for the GTIN-13: 9 310029 47737 d

(The check vector is $\mathbf{c} = [1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1]$ and calculations are performed in $\mathbb{Z}_{10}$.)

Systems of equations

5. Find the equation of the line of intersection of the planes $x + y + z = 8$ and $x - y - 3z = 2$.

6. Solve the following systems of equations by writing down the associated augmented matrix and row reducing.

(a)
$$\begin{array}{rcrcrcr} 3\,x &+& 2\,y &-& z &=& 0 \\ x &+& 4\,y &+& z &=& 8 \\ x &-& y &+& 6\,z &=& 3 \end{array}$$

(b)
$$\begin{array}{rcrcrcr} x &+& y &-& z &=& 4 \\ 2\,x &+& y &+& 3\,z &=& 1 \\ 5\,x &+& 3\,y &+& 5\,z &=& 2 \end{array}$$

(c)
$$\begin{array}{rcrcrcr} x &+& 4\,y &+& 6\,z &=& 0 \\ 2\,x &-& 2\,y &+& 3\,z &=& 0 \\ x &+& 3\,y &+& 7\,z &=& 0 \end{array}$$

(d)
$$\begin{array}{rcrcrcr} x &-& y &-& z &=& 10 \\ 2\,x &+& y &+& z &=& 5 \\ 7\,x &+& 8\,y &+& 8\,z &=& -5 \end{array}$$

Matrices

7. Let $A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 4 \\ 3 & -8 \end{bmatrix}$, $D = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$.

(a) Say which of the following products are defined, and state the size of the product for those that are defined.
A^2, AB, AC, AD, BA, B^2, BC, BD, CA, CB, C^2, CD, DA, DB, DC, D^2.

(b) Find the products that are defined.

(c) Find B^{-1} and C^{-1}.

Solutions

1. (a) $\begin{bmatrix} -7 \\ 8 \\ -40 \end{bmatrix}$ (b) 17 (c) 14 (d) $\sqrt{17}$ (e) $\sqrt{77}$ (f) $\begin{bmatrix} -8 \\ -32 \\ -5 \end{bmatrix}$

(g) $\begin{bmatrix} \frac{4}{\sqrt{17}} \\ \frac{-1}{\sqrt{17}} \\ 0 \end{bmatrix}$ (h) 1.17 radians (to 2 decimal places)

2. (a) The direction of the straight line through P and Q is $\overrightarrow{PQ} = \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$.

So a vector equation is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$, and parametric equations are $x = 2 - 5t$, $y = 1 + 3t$, $z = 2t$.

(b) A normal to the plane is given by the cross product of any two vectors in the plane. Two vectors in the plane are $\overrightarrow{PQ} = \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} -1 \\ 1 \\ -5 \end{bmatrix}$.

Now, $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} -17 \\ -27 \\ -2 \end{bmatrix}$, so this vector is normal to the plane containing the 3 points.

(c) If $X = (x, y, z)$ is an arbitrary point in the plane through the 3 points, then $\overrightarrow{PX} = \begin{bmatrix} x-2 \\ y-1 \\ z \end{bmatrix}$ is an arbitrary vector in the plane and will be perpendicular to the normal. Hence, the dot product of the normal and $\overrightarrow{PX}$ is equal to zero. That is,

$$\begin{bmatrix} -17 \\ -27 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x-2 \\ y-1 \\ z \end{bmatrix} = 0.$$

The Cartesian equation is therefore $-17(x-2) - 27(y-1) - 2z = 0$, or $17x + 27y + 2z = 61$.

3. (a) 4 (b) 6 (c) In $\mathbb{Z}_5$, $2^4 = 1$, so $2^{500} = 1$ and $2^{501} = 2$.

4. Let $\mathbf{u} = [9, 3, 1, 0, 0, 2, 9, 4, 7, 7, 3, 7, d]$. Then $\mathbf{u} \cdot \mathbf{c} = 8 + d$ in $\mathbb{Z}_{10}$, and this must equal zero in $\mathbb{Z}_{10}$, so $d = 2$.

5. We find the equation of the line of intersection by solving

$$\begin{array}{ccccccc} x & + & y & + & z & = & 8 \\ x & - & y & - & 3z & = & 2. \end{array}$$

The augmented matrix for this system reduces to $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 1 & 2 & 3 \end{array}\right]$. So the equation of the line of intersection (in parametric form) is $x = 5 + t$, $y = 3 - 2t$, $z = t$.

6. (a) $x = -1$, $y = 2$, $z = 1$.

(b) The system is inconsistent. There is no solution.

(c) $x = y = z = 0$.

(d) $x = 5$, $y = -5 - t$, $z = t$.

7. (a) The matrices which are defined are AB, AD, B^2, BD, CA, C^2.
Their sizes are 2×3, 2×1, 3×3, 3×1, 2×3, and 2×2 respectively.

(b) $AB = \begin{bmatrix} 17 & -1 & -4 \\ 14 & 3 & -6 \end{bmatrix}$, $AD = \begin{bmatrix} -11 \\ -27 \end{bmatrix}$, $B^2 = \begin{bmatrix} 7 & -3 & -1 \\ 6 & 1 & 0 \\ 3 & -3 & 4 \end{bmatrix}$,

$BD = \begin{bmatrix} 6 \\ -7 \\ 15 \end{bmatrix}$, $CA = \begin{bmatrix} -9 & 4 & 19 \\ 19 & -8 & -33 \end{bmatrix}$, $C^2 = \begin{bmatrix} 13 & -36 \\ -27 & 76 \end{bmatrix}$.

(c) $B^{-1} = \dfrac{1}{11} \begin{bmatrix} 1 & 1 & 3 \\ -6 & 5 & 4 \\ 3 & 3 & -2 \end{bmatrix}$, $C^{-1} = \begin{bmatrix} 2 & 1 \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$.