Important Ideas and Useful Facts:

(i) **Average rate of change of a function:** If a function \( y = f(x) \) is defined for \( x \) such that \( a \leq x \leq b \) then the quotient

\[
\frac{f(b) - f(a)}{b - a},
\]

represents the *average rate of change* of \( f(x) \) as \( x \) moves from \( a \) to \( b \). If \( f(x) \) denotes displacement of an object at time \( x \), then this is the average speed of the object over that time interval. This is also the slope of the secant joining endpoints of the curve.

(ii) **Limits:** Let \( y = f(x) \) be a function defined on an interval containing \( x = a \), but not necessarily defined at \( x = a \). We say that the limit of \( f(x) \) is \( L \) as \( x \) approaches \( a \), and write

\[
\lim_{x \to a} f(x) = L,
\]

if \( f(x) \) gets closer and closer to \( L \) as \( x \) gets closer and closer to \( a \).

If in fact \( f(a) \) is defined and the limit \( L \) exists and equals \( f(a) \) then we say that \( f \) is *continuous* at \( x = a \). Otherwise we say that \( f \) is *discontinuous* at \( x = a \).
(iii) **Derivative of a function:** If \( y = f(x) \) is a function defined on an interval containing \( x = a \) then the derivative of \( f \) at \( a \) is

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},
\]

and represents the instantaneous rate of change of \( f(x) \) as \( x \) moves through \( a \). If \( f(x) \) denotes displacement of an object at time \( x \), then \( f'(a) \) is the instantaneous velocity of the object when \( x = a \). This is also the slope of the tangent to the curve at \( x = a \).

(iv) **Leibniz notation and differentials:** An alternative way of expressing the derivative as a limit is the following:

\[
f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}
\]

where \( \Delta x \) represents a small change in the value of \( x \) and \( \Delta y \) as the corresponding small change in the value of \( y = f(x) \). With this notation it is common and very useful to write

\[
f'(x) = \frac{dy}{dx} \quad \text{and} \quad f'(x)dx = dy,
\]

called *Leibniz notation*, and think of \( dy \) and \( dx \) as ‘infinitesimally’ small idealised values called *differentials*. We think of the Greek \( \Delta \) becoming our ordinary \( d \) ‘in the limit’.

(v) **Some common derivatives and properties:**

(a) \( \frac{d}{dx}(k) = 0 \) for any constant \( k \), \( \frac{d}{dx}(x) = 1 \), \( \frac{d}{dx}(x^2) = 2x \), \( \frac{d}{dx}(x^n) = nx^{n-1} \).

(b) \( (kf)' = kf' \) for any constant \( k \), \( (f + g)' = f' + g' \), \( (f - g)' = f' - g' \).

(c) \( \frac{d}{dx}(\sin x) = \cos x \), \( \frac{d}{dx}(\cos x) = -\sin x \), \( \frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} = \sec^2 x \).

(d) \( \frac{d}{dx}(e^x) = e^x \), \( \frac{d}{dx}(\ln x) = \frac{1}{x} \), \( \frac{d}{dx}(\sinh x) = \cosh x \), \( \frac{d}{dx}(\cosh x) = \sinh x \).
Tutorial Exercises:

1. Graph $y = \sin x$. For which $x$-values do
   (i) the $y$-values equal (a) 0? (b) 1? (c) −1?
   (ii) the slopes of the tangent lines appear to equal (a) 0? (b) 1? (c) −1?

2. Graph $y = \cos x$. For which $x$-values do
   (i) the $y$-values equal (a) 0? (b) 1? (c) −1?
   (ii) the slopes of the tangent lines appear to equal (a) 0? (b) 1? (c) −1?

3. Use your findings from the previous two exercises as a check for what you have been told about the derivatives of $\sin x$ and $\cos x$.

4. A missile is catapulted high in the air, and its height $h$ metres above ground recorded:

   | time $t$ seconds | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
   | height $h$ metres | 3 | 30 | 47 | 54 | 51 | 38 | 15 |

   (i) Calculate the average velocity of the missile over the time interval $0 \leq t \leq 3$.
   (ii) Calculate the average velocity now for $3 \leq t \leq 6$. Interpret the negative sign.

5. Let $f(x) = -5x^2 + 32x + 3$. Check that $h = f(t)$ produces the values in the table of the previous exercise.
   (i) Find $f'(x)$ and evaluate $f'(0)$, $f'(1)$, $f'(2)$, $f'(3)$, $f'(4)$, $f'(5)$ and $f'(6)$.
   (ii) Interpret your answers in terms of the missile slowing down and speeding up.
   (iii) Find $x$ such that $f'(x) = 0$ and the maximum height reached by the missile.

6. The graph below describes the distance function $s = s(t)$ travelled by a car (in appropriate units) where $t$ denotes time. Use its shape to answer the following:

   (i) What was the initial velocity? Was the car going faster at $B$ or at $C$?
   (ii) Was the car slowing down or speeding up at each of $A$, $B$ and $C$?
   (iii) What appears to have happened at $D$ and $E$ and in between?
Further Exercises:

∗7. An arrow is shot upwards on the moon with a velocity of 58 m/sec and its height \( H \) metres after \( t \) seconds is given by the formula \( H = 58t - 0.83t^2 \).

(i) Find a formula for the velocity and evaluate it at \( t = 1 \).
(ii) Find the moment when the arrow is instantaneously at rest, and calculate its height above the moon.
(iii) At what time will the arrow return to the moon and at what velocity?

∗8. Verify that \( \frac{1}{x+h} \frac{1}{x} \rightarrow -\frac{1}{x^2} \) as \( h \rightarrow 0 \). Deduce that if \( y = x^{-1} \) then \( dy/dx = -x^{-2} \).

∗9. According to Boyle’s Law, if the temperature of a confined gas is held fixed, then the product of the pressure \( P \) and the volume \( V \) is constant. Suppose that for a certain gas \( PV = 500 \) where \( P \) is measured in newtons per square metre and \( V \) is measured in cubic metres. Express \( V \) as a function of \( P \) and use the result of the previous exercise to find a formula for the instantaneous rate of change of \( V \) with respect to \( P \). What happens if there is no pressure?

**10. Verify that \( \frac{\sqrt{x+h} - \sqrt{x}}{h} \rightarrow \frac{1}{2\sqrt{x}} \) as \( h \rightarrow 0 \). Deduce that if \( y = x^{1/2} \) then \( dy/dx = \frac{1}{2}x^{-1/2} \).

Short Answers to Selected Exercises:

1. (i) (a) \( n\pi \) (b) \( 2n\pi + \pi/2 \) (c) \( 2n\pi - \pi/2 \) (ii) (a) \( 2n\pi \pm \pi/2 \) (b) \( n\pi \), \( n \) even (c) \( n\pi \), \( n \) odd
2. (i) (a) \( 2n\pi \pm \pi/2 \) (b) \( n\pi \), \( n \) even (c) \( n\pi \), \( n \) odd (ii) (a) \( n\pi \) (b) \( 2n\pi - \pi/2 \) (c) \( 2n\pi + \pi/2 \)
4. (i) 17 m/sec (ii) −13 m/sec, missile is falling (upwards being the positive direction)
5. (i) −10\(x + 32\), 32, 22, 12, 2, −8, −18, −28 (iii) 3.2, 54.2 m
6. (i) zero, faster at \( C \) (ii) speeding up at \( A \), \( C \); slowing down at \( B \) (iii) stopping at \( D \) and reversing till \( E \), then moving forwards
7. (i) 58 − 1.66\(t \) m/sec, 56.34 m/sec (ii) 34.94 sec, 1,013 m (iii) 69.88 sec, −58 m/sec
9. −500/\(P^2\)