The University of Sydney

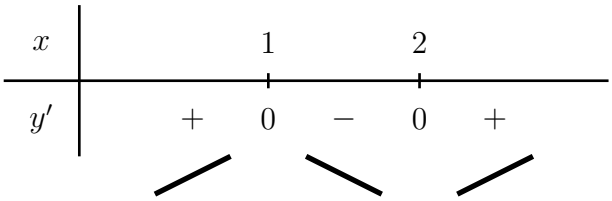
MATH1111 Introduction to Calculus

Semester 1 **Longer Solutions to Week 10 Exercises (Thurs/Fri)** 2017

1. We have

$$y' \;=\; 6x^2 - 18x + 12 \;=\; 6(x^2 - 3x + 2) \;=\; 6(x-1)(x-2)\ ,$$

so $y' = 0$ when $x = 1$ or $x = 2$. The sign diagram for y' is



so that a local maximum occurs when $x = 1$, and a local minimum when $x = 2$. When $x = 1$, we have

$$y \;=\; 2(1)^3 - 9(1)^2 + 12(1) - 1 \;=\; 4\ ,$$

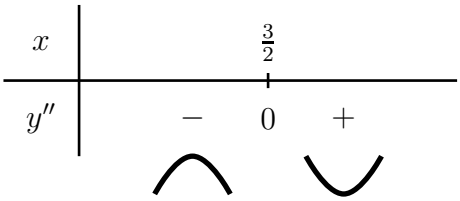
and when $x = 2$, we have

$$y \;=\; 2(2)^3 - 9(2)^2 + 12(2) - 1 \;=\; 3\ .$$

The turning points on the graph are therefore $(1, 4)$ and $(2, 3)$. Differentiating again, gives

$$y'' \;=\; 12x - 18 \;=\; 6(2x - 3)\ ,$$

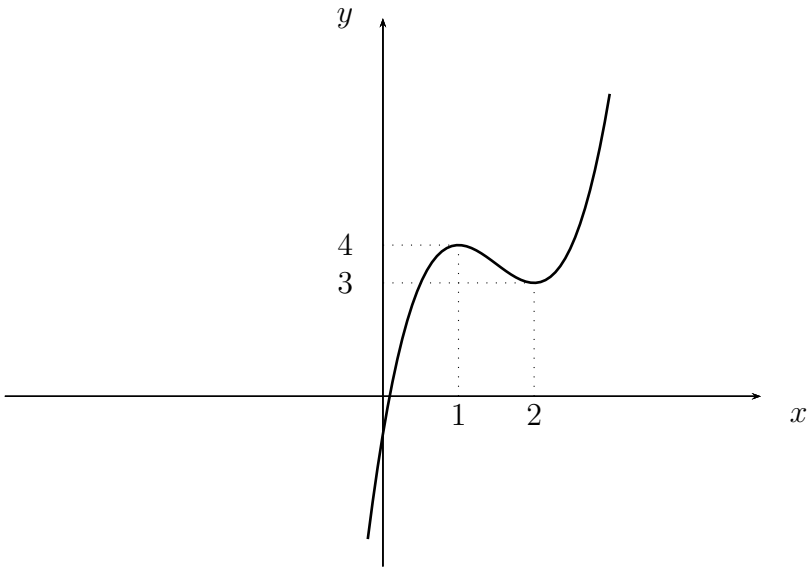
so $y'' = 0$ when $2x - 3 = 0$, that is, $x = 3/2$. The sign diagram for y'' is



so that change of concavity takes place when $x = 3/2$, giving

$$y \;=\; 2(3/2)^3 - 9(3/2)^2 + 12(3/2) - 1 \;=\; \frac{27}{4} - \frac{81}{4} + 18 - 1 \;=\; \frac{7}{2}\ .$$

The point of inflection on the graph is therefore $(\frac{3}{2}, \frac{7}{2})$.



2. Differentiating yields

$$y' \;=\; 6x^2 - 6x - 12 \;=\; 6(x^2 - x - 2) \;=\; 6(x+1)(x-2)\ ,$$

so that $y' = 0$ when $x = -1$ and $x = 2$. We look at y-values at these critical points and the end points of the interval $-2 \le x \le 3$:

$$y(-2) = -16 - 12 + 24 + 1 = -3\,,\ \ y(-1) = -2 - 3 + 12 + 1 = 8\,,$$

$$y(2) = 16 - 12 - 24 + 1 = -19\,,\ \ y(3) = 54 - 27 - 36 + 1 = -8\ ,$$

so the absolute maximum achieved by y is 8 and the absolute minimum -19.

3. Differentiating, using the Quotient Rule, yields

$$y' \;=\; \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} \;=\; \frac{1 - x^2}{(x^2+1)^2} \;=\; \frac{(1-x)(1+x)}{(x^2+1)^2}\ ,$$

so that $y' = 0$ when $x = -1$ and $x = 1$. Evaluating at end points of the interval $-1 \le x \le 2$ and the critical point $x = 1$ inside the interval:

$$y(0) = \frac{0}{0+1} = 0\,,\quad y(1) = \frac{1}{1+1} = \frac{1}{2}\,,\quad y(2) = \frac{2}{4+1} = \frac{2}{5}\ ,$$

so the absolute maximum achieved by y is $1/2$ and the absolute minimum 0.

4. (i) $f'(x) = 5x^4 - 5 = 5(x^4 - 1) = 5(x^2+1)(x-1)(x+1)\ ,\quad f''(x) = 20x^3\ .$

(ii) $f'(x) = \dfrac{1(x^2+4) - x(2x)}{(x^2+4)^2} = \dfrac{4 - x^2}{(x^2+4)^2} = \dfrac{(2-x)(2+x)}{(x^2+4)^2}\ ,$

$f''(x) = \dfrac{(-2x)(x^2+4)^2 - (4-x^2)(2)(x^2+4)(2x)}{(x^2+4)^4} = \dfrac{2x(x^2-12)}{(x^2+4)^3}\ .$

(iii) $f(x) = x + (1-x)^{1/2}\ ,\quad f'(x) = 1 + \frac{1}{2}(1-x)^{-1/2}(-1) = 1 - \frac{1}{2}(1-x)^{-1/2}$

$f''(x) = (-\dfrac{1}{2})(-\dfrac{1}{2})(1-x)^{-3/2}(-1) \;=\; \dfrac{-1}{4(1-x)^{3/2}}\ .$

5. (i) We have $y = f(x) = x^5 - 5x + 3$ and $y' = f'(x) = 5(x^2+1)(x-1)(x+1)$, so the sign diagram for y' is

x		-1		1	
y'	$+$	0	$-$	0	$+$

Hence the function is increasing for $x \le -1$ and $x \ge 1$ (graph sloping upwards) and decreasing for $-1 \le x \le 1$ (graph sloping downwards). The turning points therefore are

$$(-1, f(-1)) \;=\; (-1, 7)\;,$$

which is a local maximum, and

$$(1, f(1)) \;=\; (1, -1)\;,$$

which is a local minimum.

(ii) We have $y = f(x) = \dfrac{x}{x^2+4}$ and $y' = f'(x) = \dfrac{(2-x)(2+x)}{(x^2+4)^2}$, so the sign diagram for y' is

x		-2		2	
y'	$-$	0	$+$	0	$-$

Hence the function is increasing for $-2 \le x \le 2$ (graph sloping upwards) and decreasing for $x \le -2$ and $x \ge 2$ (graph sloping downwards). The turning points therefore are

$$(-2, f(-2)) \;=\; (-2, -1/4)\;,$$

which is a local (and in fact global) minimum, and

$$(2, f(2)) \;=\; (2, 1/4)\;,$$

which is a local (and in fact global) maximum.

(iii) We have $y = f(x) = x + \sqrt{1-x}$. Note that this is a sensibly defined function only for $x \le 1$ (because we can't take square roots of negative numbers). We have

$$y' = f'(x) = 1 - \frac{1}{2}(1-x)^{-1/2} = 1 - \frac{1}{2\sqrt{1-x}} = \frac{2\sqrt{1-x}-1}{2\sqrt{1-x}}\;.$$

Then $y' = 0$ when

$$2\sqrt{1-x} \;=\; 1\;,$$

that is, $4(1-x) = 1$, so $1 - x = 1/4$, yielding $x = 3/4$. The sign diagram for y' is

x		$\frac{3}{4}$	
y'	$+$	0	$-$

Hence the function is increasing for $x \leq \frac{3}{4}$ (graph sloping upwards) and decreasing for $\frac{3}{4} \leq x \leq 1$ (graph sloping downwards). The turning point therefore is

$$(3/4, f(3/4)) \;=\; (3/4, 5/4)\,,$$

which is a global maximum.

6. (i) We have $y'' = f''(x) = 20x^3$, so the sign diagram for y'' is

x		0	
y''	$-$	0	$+$

so there is just one point of inflection, which is $(0, f(0)) = (0, 3)$.

(ii) We have

$$y'' \;=\; f''(x) \;=\; \frac{2x(x^2-12)}{(x^2+4)^3} \;=\; \frac{2x(x-2\sqrt{3})(x+2\sqrt{3})}{(x^2+4)^3}\,,$$

so the sign diagram for y'' is

x		$-2\sqrt{3}$		0		$2\sqrt{3}$	
y''	$-$	0	$+$	0	$-$	0	$+$

so there are three points of inflection, namely $(-2\sqrt{3}, f(-2\sqrt{3})) = (-2\sqrt{3}, -\sqrt{3}/8)$, $(0, f(0) = (0, 0)$ and $(2\sqrt{3}, f(2\sqrt{3})) = (2\sqrt{3}, \sqrt{3}/8)$.

(iii) In this case

$$y'' \;=\; f''(x) \;=\; \frac{-1}{4(1-x)^{3/2}}\,,$$

which is always negative where it is defined, so the curve is concave down and there are no points of inflection.

7. Differentiating yields

$$y' \;=\; 3(x^2-1)^2(2x) \;=\; 6x\big((x-1)(x+1)\big)^2 \;=\; 6x(x-1)^2(x+1)^2\,,$$

so that $y' = 0$ when $x = 0, -1, 1$. Evaluating at these critical points and at end points of the interval $-1 \leq x \leq 2$:

$$y(-1) = 0^3 = 0\,, \quad y(0) = (-1)^3 = -1\,, \quad y(1) = 0^3 = 0\,, \quad y(2) = 3^3 = 27\,,$$

so the absolute maximum achieved by y is 27 and the absolute minimum -1.

8. We have $f(x) = 5 - 3x^2 + x^3$ so

$$f'(x) \;=\; -6x + 3x^2 \;=\; 3x(x-2)$$

and

$$f''(x) \;=\; -6 + 6x \;=\; 6(x-1)\,.$$

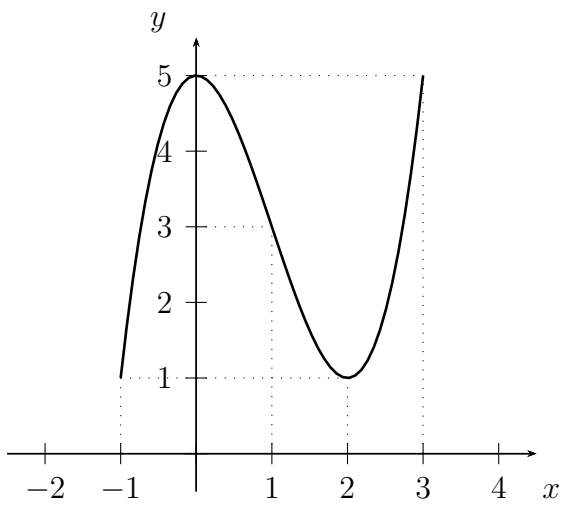
Then $f'(x) = 0$ when $x = 0$ and $x = 2$. But $f''(0) = -6 < 0$, so, by the Second Derivative Test, there is a local maximum at $x = 0$, taking the value $f(0) = 5$. Also, $f''(2) = 6 > 0$, so there is a local minimum at $x = 2$, taking the value $f(2) = 1$. These results are confirmed by looking at the sign diagram for the first derivative:

x		0		2	
$f'(x)$	$+$	0	$-$	0	$+$

The sign diagram for the second derivative

x		1	
$f''(x)$	$-$	0	$+$

tells us that an inflection (change in concavity) takes place at $x = 1$. The following is a sketch of the graph of $y = f(x)$ for $-1 \le x \le 3$:



9. Let x cm be the width and y cm the height of the poster, so $xy = 180$. We want to maximise the printing area

$$A \;=\; (x-2)(y-3) \;=\; (x-2)\left(\frac{180}{x} - 3\right) \;=\; 186 - 3x - \frac{360}{x}\ .$$

This occurs when $A' = -3 + \frac{360}{x^2} = 0$, that is, $x^2 = 120$, so that $x = \sqrt{120} \approx 10.95$. Hence we choose the poster to be approximately 11 cm wide (and about 16.4 cm high).

10. We have $v = a(R-r)r^2 = aRr^2 - ar^3$ so that

$$\frac{dv}{dr} \;=\; 2aRr - 3ar^2 \;=\; ar(2R - 3r)\ ,$$

which is zero when $r = 0$ or $r = 2R/3$. The sign diagram for dv/dr is

x		0		$\frac{2R}{3}$	
$\frac{dv}{dr}$	$-$	0	$+$	0	$-$

so that a maximum coughing speed occurs when $r = 2R/3$ (since we may ignore $r < 0$).