MATH 1111 Week 10 (Mon/Tues) Worksheet

1. Let $y = 4x^2 - 4x + 3$.

Find $y' = \boxed{8x - 4}$.

For which x is $y' = 0$?

Answer: $x = \boxed{\frac{4}{8} = \frac{1}{2}}$

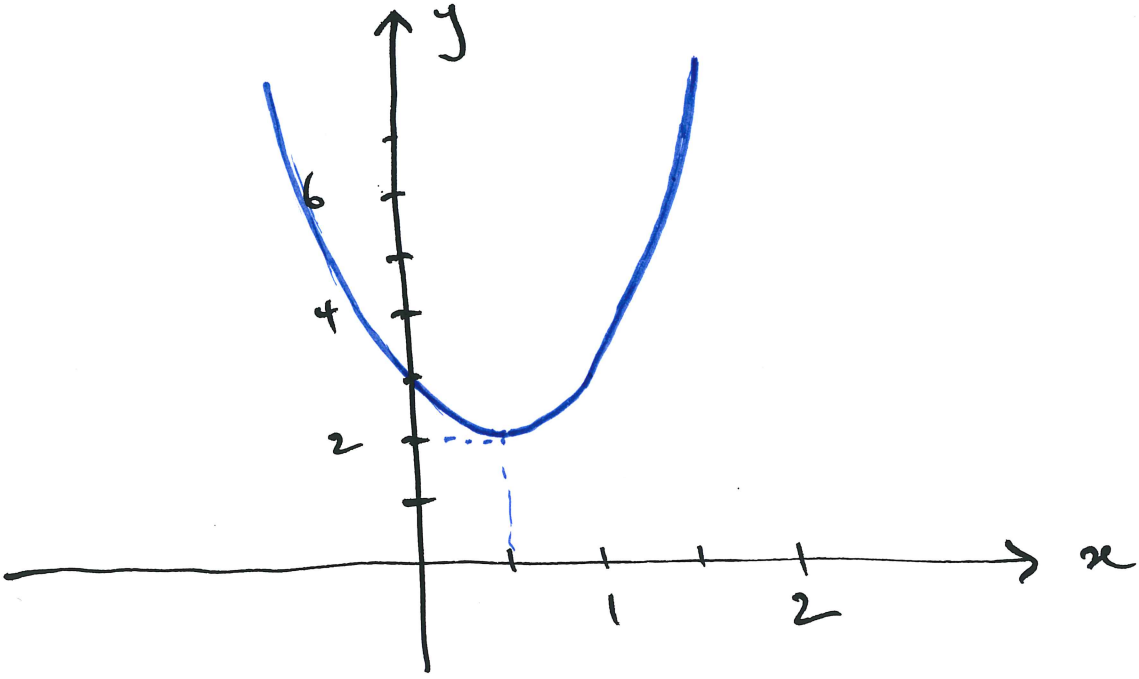
Draw a sign diagram for y':

x		$\frac{1}{2}$	
y'	$-$	0	$+$

The minimum value for y is

$$\boxed{y(\tfrac{1}{2}) = 1 - 2 + 3 = 2}$$

Sketch the parabola $y = 4x^2 - 4x + 3$:



2. Approach the quadratic from the previous exercise by completing the square:

$$y = 4x^2 - 4x + 3$$
$$= (4x^2 - 4x + 1) + 2$$
$$= (2x-1)^2 + 2 \qquad (*)$$

Put $u = 2x-1$, so $y = u^2 + 2$.

then $\frac{du}{dx} = \boxed{2}$ and $\frac{dy}{du} = \boxed{2u}$.

By the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \boxed{(2u)(2) = 4u = 8x - 4}$$

as before. ✓

Use (*) to explain why the minimum value of y must be 2 when $x = \frac{1}{2}$, without using the derivative:

$y = (2x-1)^2 + 2 \geqslant 2$ because $(2x-1)^2 \geqslant 0$ always, and $y = 2$ when $2x-1 = 0$, i.e. $x = \frac{1}{2}$.

3. Let

$$y = 2x^3 - 9x^2 + 12x - 3 .$$

Find

$$y' = \boxed{\begin{aligned} &6x^2 - 18x + 12 \\ &= 6(x^2 - 3x + 2) \\ &= 6(x-2)(x-1) . \end{aligned}}$$

and

$$y'' = \boxed{\begin{aligned} &12x - 18 \\ &= 6(2x - 3) \end{aligned}}$$

Then $y' = 0$ when $x = \boxed{1}$ and $x = \boxed{2}$

and $y'' = 0$ when $x = \boxed{\frac{3}{2}}$.

Draw sign diagrams for y' and y'' :

x		1		2	
y'	+	0	−	0	+
	↗		↘		↗

x		$3/2$	
y''	−	0	+
	∩		∪

3. (continued)

The local maximum occurs when $x = \boxed{1}$

and $y = \boxed{2 - 9 + 12 - 3 = 14 - 12 = 2}$

The local minimum occurs when $x = \boxed{2}$

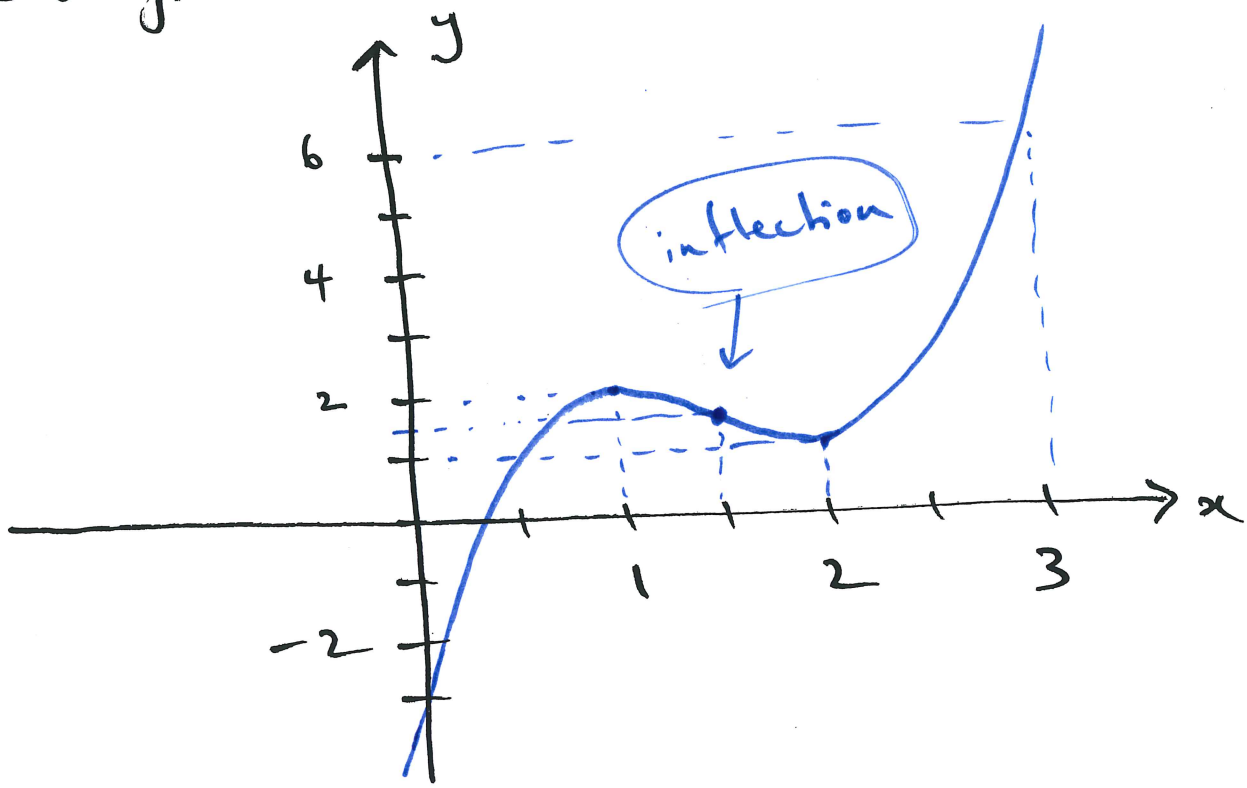
and $y = \boxed{\begin{aligned} &16 - 36 + 24 - 3 \\ &= 40 - 39 = 1 \end{aligned}}$

The inflection occurs when $x = \boxed{3/2}$

and $y = \boxed{\begin{aligned} &2\left(\tfrac{27}{8}\right) - 9\left(\tfrac{9}{4}\right) + 18 - 3 \\ &= \frac{27 - 81}{4} + 15 = -\frac{27}{2} + 15 = 3/2 \end{aligned}}$

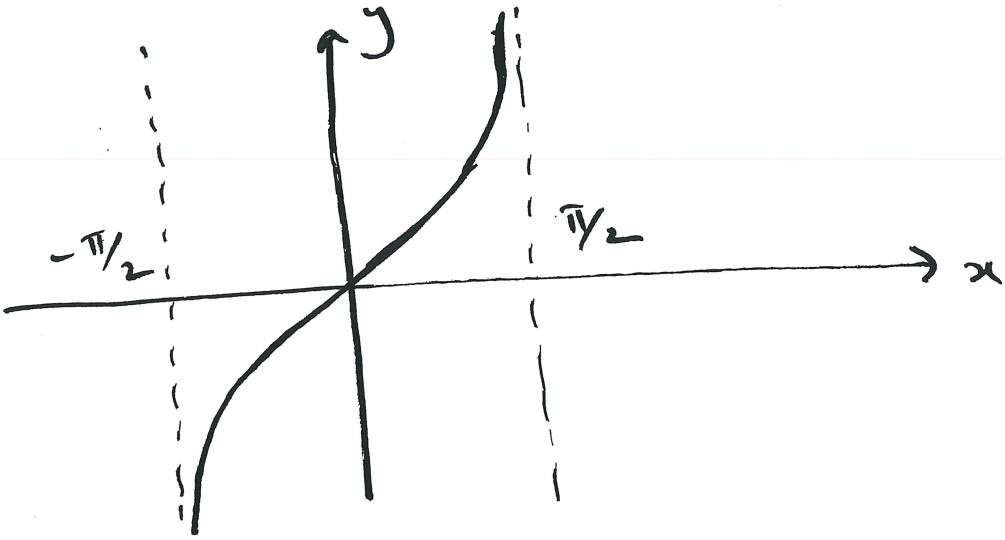
Add the curve $y = 2x^3 - 9x^2 + 12x - 3$

to the diagram:



4. Consider the curve

$$y = \tan x = \frac{\sin x}{\cos x} \quad \text{for } -\pi/2 < x < \frac{\pi}{2}:$$



(i) Is the function increasing or decreasing?

Answer: $\boxed{\text{increasing}}$

(ii) Where is the curve concave up?

Answer: $\boxed{\text{for } x \text{ where } 0 < x < \pi/2}$

(iii) Where is the curve concave down?

Answer: $\boxed{\text{for } x \text{ where } -\frac{\pi}{2} < x < 0}$

(iv) The inflection point is $(x, y) = \boxed{(0,0)}$

(v) Complete the following:

$$\lim_{x \to \pi/2^-} \tan x = \boxed{\infty}, \quad \lim_{x \to -\pi/2^+} \tan x = \boxed{-\infty}$$

5. Let $y = \tan x = \dfrac{u}{v}$

where $u = \sin x$ and $v = \cos x$.

Then $\dfrac{du}{dx} = \boxed{\cos x}$, $\dfrac{dv}{dx} = \boxed{-\sin x}$

so $y' = \dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$= \boxed{\frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}}$$

Use the Chain Rule to find

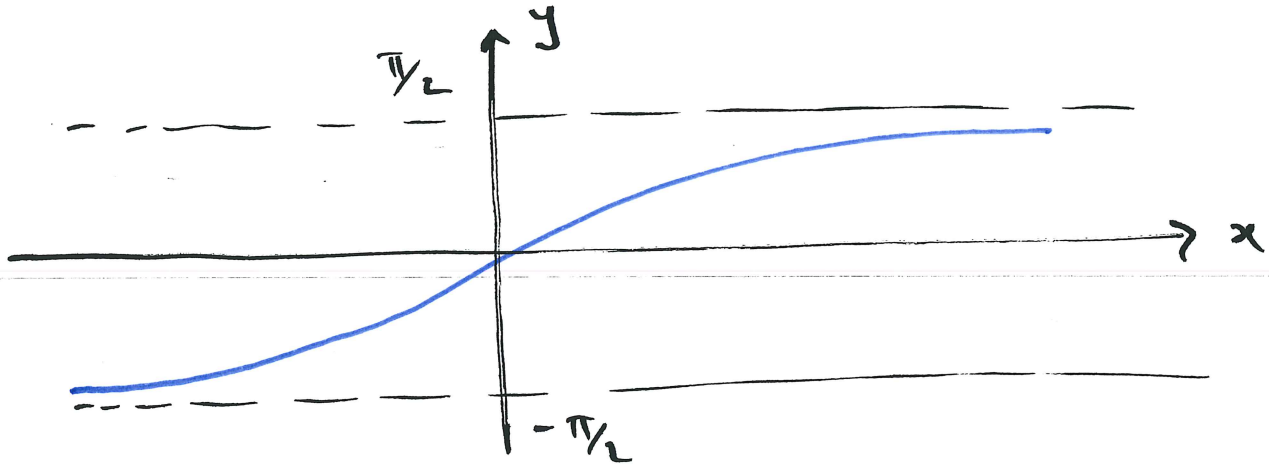
$$y'' = \boxed{\begin{aligned} &\frac{d}{dx}\left(\frac{1}{\cos^2 x}\right) = \frac{d}{dx}\left(\cos^{-2} x\right) \\ &= -2\left(\cos^{-3} x\right)(-\sin x) = \frac{2\sin x}{\cos^3 x} \end{aligned}}$$

Complete the sign diagram for y'' near $x = 0$

x		0	
y''	$-$	0	$+$
	concave down		concave up

(confirming that $(0,0)$ is the inflection point of the curve $y = \tan x$).

6. Add the curve $y = \tan^{-1} x$ below:



(i) Is the function increasing or decreasing?

Answer: increasing

(ii) Where is the curve concave up?

Answer: for $x < 0$

(iii) Where is the curve concave down?

Answer: for $x > 0$

(iv) The inflection point is $(x, y) =$ $(0, 0)$

(v) Complete the following:

$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$, $\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

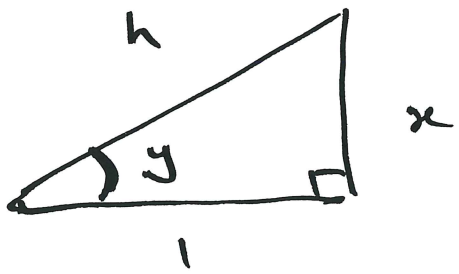
7. Let $y = \tan^{-1} x$, so $x = \tan y$.

Then $\dfrac{dx}{dy} = \boxed{\dfrac{1}{\cos^2 y}}$ (in terms of y)

Hence $\dfrac{dy}{dx} = \dfrac{1}{dx/dy} = \boxed{\cos^2 y}$

(still in terms of y)

Consider the following right-angled triangle:



Express the following in terms of x:

$\tan y = \boxed{x}$, $h = \boxed{\sqrt{1+x^2}}$

$\sin y = \boxed{\dfrac{x}{\sqrt{1+x^2}}}$, $\cos y = \boxed{\dfrac{1}{\sqrt{1+x^2}}}$

Now find $\dfrac{dy}{dx}$ in terms of x:

$$\frac{dy}{dx} = \boxed{\cos^2 y = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}}.$$

(and you obtain the witch of Maria Agnesi!!)