THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorials Weeks 12 and 13

MATH1111: Introduction to Calculus Semester 1, 2009

Web Page: `http://www.maths.usyd.edu.au/u/UG/JM/MATH1111/`
Lecturer: Clio Cresswell

Textbook questions refer to *Calculus: Single and Multivariable*, by Deborah Hughes-Hallett, Andrew M. Gleason, William G. McCallum *et al.*, John Wiley & Sons, 4th ed.

1. Section 14.1: 2, 8, 18.

Solution:

2. Using $\Delta x = 0.1$ and $\Delta y = 0.1$, we have the estimates $f_x(1,3) \approx \frac{f(1.1,3)-f(1,3)}{0.1} \approx \frac{0.0470-0.0519}{0.1} = -0.0493$ and $f_y(1,3) \approx \frac{f(1,3.1)-f(1,3)}{0.1} \approx \frac{0.0153-0.0519}{0.1} = -0.3660$. Now using $\Delta x = 0.01$ and $\Delta y = 0.01$, we have the estimates $f_x(1,3) \approx \frac{f(1.01,3)-f(1,3)}{0.01} \approx \frac{0.0514-0.0519}{0.01} = -0.0501$ and $f_y(1,3) \approx \frac{f(1,3.01)-f(1,3)}{0.01} \approx \frac{0.0483-0.0519}{0.01} = -0.3629$.

8. (a) At Q, R, we have $f_x < 0$ because f decreases as we move in the x-direction.
(b) At Q, P, we have $f_y > 0$ because f increases as we move in the y-direction.
(c) At all four points, P, Q, R, S, we have $f_{xx} > 0$, because f_x is incresing as we move in the x-direction. (At P, S, we see that f_x is positive and getting larger; at Q, R, we see that f_x is negative and getting less negative.)
(d) At all four points, P, Q, R, S, we have $f_{yy} > 0$, so there are none with $f_{yy} < 0$. The reasoning is similar to part (c).

18. (a) For points near the point $(0, 5, 3)$, moving in the positive x-direction, the surface is sloping down and the function is decreasing. Thus $f_x(0, 5) < 0$.
(b) Moving in the positive y-direction near this point the surface slopes up as the function increases, so $f_y(0, 5) > 0$.

2. Section 14.2: 1, 2, 4, 7, 8, 10, 16, 18, 22, 25, 26, 34, 41.

Solution:

1. $f_x(x, y) = 10xy^3 + 8y^2 - 6x$ and $f_y(x, y) = 15x^2y^2 + 16xy$.

2. $f_x(x, y) = 3x^2 + 6xy$ and $f_y(x, y) = 3x^2 - 4y$, so $f_x(1, 2) = 15$ and $f_y(1, 2) = -5$.

4. $\frac{\partial z}{\partial x} = 7(x^2 + x - y)^6(2x + 1)$ and $\frac{\partial z}{\partial y} = -7(x^2 + x - y)^6$.

7. Using the quotient rule $z_y = \frac{(15xy-8)(21x^2y^6-2y)-15x(3x^2y^7-y^2)}{(15xy-8)^2}$.

8. $\frac{\partial}{\partial T}\left(\frac{2\pi r}{T}\right) = -\frac{2\pi r}{T^2}$.

10. $\frac{\partial}{\partial x}(xe^{\sqrt{xy}}) = e^{\sqrt{xy}} + xe^{\sqrt{xy}} \times \frac{1}{2}(xy)^{-1/2}y = e^{\sqrt{xy}}(1 + \frac{\sqrt{xy}}{2})$.

16. $\frac{\partial}{\partial r}\left(\frac{2\pi r}{v}\right) = \frac{2\pi}{v}$.

18. $\frac{\partial}{\partial v_0}(v_0 + at) = 1$.

22. $\frac{\partial f_0}{\partial L} = \frac{1}{2\pi}(-\frac{1}{2})(LC)^{-3/2}C = -\frac{1}{4\pi L\sqrt{LC}}$.

25. $f_a = e^a \sin(a + b) + e^a \cos(a + b)$.

26. $z_x = \cos(5x^3y - 3xy^2) \times (15x^2y - 3y^2)$.

34. $\frac{\partial z}{\partial y} = 2e^{x+2y} \sin y + e^{x+2y} \cos y$ so at $(1, 0.5)$ have $\frac{\partial z}{\partial y} = 13.6$.

41. For a 70 kg person at the surface of the earth, we have $F = \frac{(6.67\times10^{-11})(6\times10^{24})(70)}{(6.4\times10^6)^2} \approx 684$. So the gravitational force on this person is about 684 newtons.

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Differentiating gives $\frac{\partial F}{\partial m} = \frac{GM}{r^2}$ and for our person $\frac{\partial F}{\partial m} \approx 9.77$ newtons/kg. And also $\frac{\partial F}{\partial r} = \frac{-2GMm}{r^3}$ and for our person $\frac{\partial F}{\partial r} \approx -0.000214$ newtons/meter. These partial derivatives tell us that the gravitational force increases by about 9.77 newtons for an increase of 1 kg in the mass, and the gravitational force decreases by about 0.000214 newtons if the distance from the centre of the earth increases by 1 meter.

3. Section 14.3: 1, 2, 3, 4, 5, 6, 10, 11, 13, 18, 25.

Solution:

1. $z_x = x$ and $z_y = 4y$ so $z_x(2,1) = 2$ and $z_y(2,1) = 4$ with $z(2,1) = 4$. So the tangent plane at $(2,1,4)$ has equation $z = z(2,1) + z_x(2,1)(x-2) + z_y(2,1)(y-1) = 4 + 2(x-2) + 4(y-1)$ or $z = -4 + 2x + 4y$.

2. $z_x = e^{x/y}$ and $z_y = -\frac{x}{y}e^{x/y} + e^{x/y}$ so $z_x(1,1) = e$ and $z_y(1,1) = 0$ with $z(1,1) = e$. So the tangent plane at $(1,1,e)$ has equation $z = z(1,1) + z_x(1,1)(x-1) + z_y(1,1)(y-1) = e + e(x-1) + 0(y-1)$ or $z = ex$.

3. $z_x = 2x + 1$ and $z_y = e^y$ so $z_x(1,0) = 3$ and $z_y(1,0) = 1$ with $z(1,0) = 9$. So the tangent plane at $(1,0,9)$ has equation $z = 9 + 3(x-1) + (y-0)$ or $z = 6 + 3x + y$.

4. $z_x = \frac{2x}{x^2+1}$ and $z_y = 2y$ so $z_x(0,3) = 0$ and $z_y(0,3) = 6$ with $z(0,3) = 9$. So the tangent plane at $(0,3,9)$ has equation $z = 9 + 0(x-0) + 6(y-3)$ or $z = 6y - 9$.

5. Since $g_u = 2u + v$ and $g_v = u$, we have $dg = (2u+v)du + udv$.

6. $df = y\cos(xy)dx + x\cos(xy)dy$.

10. We have $dF = F_m dm + F_r dr$. Now $F_m = \frac{G}{r^2}$ and $F_m(100,10) = \frac{G}{100} = 0.01G$. And $F_r = -\frac{2Gm}{r^3}$ with $F_r(100,10) = -\frac{2G100}{10^3} = -0.2G$. So $dF = 0.01Gdm - 0.2Gdr$.

11. We have $dg = g_x dx + g_t dt$. Finding the partial derivatives, we have $g_x = 2x\sin(2t)$ so $g_x(2,\frac{\pi}{4}) = 4\sin(\frac{\pi}{2}) = 4$, and $g_t = 2x^2\cos(2t)$ so $g_t(2,\frac{\pi}{4}) = 8\cos(\frac{\pi}{2}) = 0$. So $dg = 4dx$.

13. $f_x = 2xy$ and $f_y = x^2$ so $f_x(3,1) = 6$ and $f_y(3,1) = 9$ with $f(3,1) = 9$. So the local linearization is $z = 9 + 6(x-3) + 9(y-1)$.

18. Local linearization gives the approximation $T(x,y) \approx T(2,1) + T_x(2,1)(x-2) + T_y(2,1)(y-1)$ or $T(x,y) \approx 135 + 16(x-2) - 15(y-1)$. Thus $T(2.04, 0.97) \approx 135 + 16(2.04-2) - 15(0.97-1) = 136.09$°C.

25. The error in η is approximated by $d\eta$, where $d\eta = \frac{\partial \eta}{\partial r}dr + \frac{\partial \eta}{\partial p}dp$. Now $\frac{\partial \eta}{\partial r} = \frac{\pi}{8}\frac{p4r^3}{v}$ and $\frac{\partial \eta}{\partial p} = \frac{\pi}{8}\frac{r^4}{v}$. For $r = 0.005$ and $p = 10^5$ we have $\frac{\partial \eta}{\partial r} = 3.14159 \cdot 10^7$ and $\frac{\partial \eta}{\partial p} = 0.39270$. So, $d\eta = \frac{\partial \eta}{\partial r}dr + \frac{\partial \eta}{\partial p}dp$ is largest when $d\eta = 3.14159 \cdot 10^7 \cdot 0.00025 + 0.39270 \cdot 1000 = 8246.68$.