

Solutions to Tutorials 27/04/09,28/04/09 and 30/04/09,01/05/09

MATH1111: Introduction to Calculus

Semester 1, 2009

Web Page: <http://www.maths.usyd.edu.au/u/UG/JM/MATH1111/>

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Textbook questions refer to *Calculus: Single and Multivariable*, by Deborah Hughes-Hallett, Andrew M. Gleason, William G. McCallum *et al.*, John Wiley & Sons, 4th ed.

1. Section 4.3: 4, 6, 19, 20, 21, 24, 28.

Solution:

4. (a) $f'(x) = 10x^9 - 10 = 10(x^9 - 1)$. $f'(x) = 0 \Rightarrow x = 1$. When $x < 1$, $f'(x) < 0$, and when $x > 1$, $f'(x) > 0$. Hence $(1, -9)$ is a local minimum. At the endpoints we have $f(0) = 0$ and $f(2) = 1004$. Since there are no other critical points $(0, 0)$ and $(2, 1004)$ are local maxima.

(b) $(1, -9)$ is a global minimum. $(2, 1004)$ is a global maximum.

6. (a) $f'(x) = 2 \sin x \cos x + \sin x = \sin x(2 \cos x + 1)$. $f'(x) = 0 \Rightarrow \sin x = 0$ or $2 \cos x + 1 = 0$. $\sin x = 0 \Rightarrow x = 0, \pi$ (only solutions on given interval). $2 \cos x + 1 = 0 \Rightarrow \cos x = -1/2$ or $x = 2\pi/3$ (again, only solution on given interval). Now $f''(x) = \cos x(2 \cos x + 1) + \sin x(-2 \sin x) = 2 \cos^2 x - 2 \sin^2 x + \cos x$. So $f''(0) > 0$, $f''(\pi) > 0$, and $f''(2\pi/3) < 0$. And $(0, -1)$ and $(\pi, 1)$ are local minima. And $(2\pi/3, 5/4)$ is a local maximum. The endpoints are $(0, -1)$ and $(\pi, 1)$.

(b) $(0, -1)$ is a global minimum. $(2\pi/3, 5/4)$ is a global maximum.

19. We want to maximize height, or y . $y' = -32t + 50$, $y' = 0 \Rightarrow t \approx 1.56$. $y'' = -32 < 0$ so this is indeed a maximum. So its maximum height (that happens after 1.56 s) is $y \approx -16(1.56)^2 + 50(1.56) + 5 = 44.1$ feet.

20. $v = aRr^2 - ar^3$, $v' = 2aRr - 3ar^2 = ar(2R - 3r)$ and $v' = 0 \Rightarrow r = 0, 2R/3$. $v'' = 2aR - 6ar$. At $r = 0$, $v'' > 0$, so $r = 0$ gives a local minimum radius. At $r = 2R/3$, $v'' = 2aR - 4aR = -2aR < 0$, so $r = 2R/3$ gives a local maximum $v = aR(2R/3)^2 - a(2R/3)^3 = 4aR^3/27$. At the endpoint $r = R$, $v = aR^3 - aR^3 = 0$. So $r = 2R/3$ maximizes the speed.

21. (a) $T' = -\frac{1}{3}D^2 + (\frac{C}{2} - \frac{D}{3})2D = -D^2 + CD = D(C - D)$. $T' = 0 \Rightarrow D = 0, C$. $T'' = -2D + C$. So when $D = 0$, $T'' > 0$ and we have a local minimum. And when $D = C$, $T'' < 0$ and we have a local maximum. So $D = C$ maximizes T .

(b) The sensitivity is $T' = -D^2 + CD$. To maximize T' consider $T'' = 0$. This gives $D = C/2$. $T''' = -2 < 0$ so $D = C/2$ is the dose that maximizes sensitivity.

24. $f'(r) = \frac{2A}{r^3} - \frac{3B}{r^4}$. $f'(r) = 0 \Rightarrow \frac{2Ar - 3B}{r^4} = 0$ or $r = \frac{3B}{2A}$. For $r < \frac{3B}{2A}$, $f'(r) < 0$. And for $r > \frac{3B}{2A}$, $f'(r) > 0$. Thus, the force between the atoms is minimized at $r = \frac{3B}{2A}$.

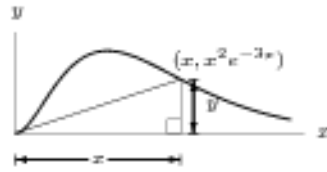
28. Let the slope be S . Then $S = \frac{y-0}{x-0} = \frac{x^2 e^{-3x}}{x} = x e^{-3x}$. $S' = e^{-3x} - 3x e^{-3x} = (1 - 3x)e^{-3x}$. $S' = 0 \Rightarrow x = 1/3$. For $x < 1/3$, $S' > 0$, and for $x > 1/3$, $S' < 0$, so $x = 1/3$ maximizes the slope and the maximum slope is $S = e^{-1}/3$.

2. Section 4.5: 11, 12, 17, 18, 19, 32.

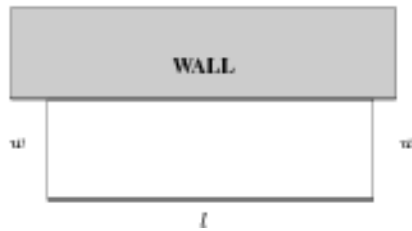
Solution:

11. $A = \text{height} \times \text{width} = \sqrt{a}(9 - a) = 9a^{1/2} - a^{3/2}$ where $0 \leq a \leq 9$. And our aim is to find the a that will maximize A . $A' = \frac{9}{2}a^{-1/2} - \frac{3}{2}a^{1/2}$. When $A' = 0$, $\frac{9}{2\sqrt{a}} = \frac{3\sqrt{a}}{2}$ or $a = 3$. $A'' = -\frac{9}{4}a^{-3/2} - \frac{3}{4}a^{-1/2}$ and when $a = 3$ $A'' < 0$ so have a local maximum. At the endpoints when $a = 0$ and when $a = 9$, $A = 0$. So $a = 3$ is not just a local maximum but also a global maximum on our interval. So the dimensions of the rectangle with maximum possible area are 6 by $\sqrt{3}$.

12. The triangle in the figure below has area, A , given by $A = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}xy = \frac{1}{2}x^3e^{-3x}$. $A' = \frac{3}{2}x^2e^{-3x} - \frac{3}{2}x^3e^{-3x} = \frac{3}{2}x^2(1-x)e^{-3x}$. When $A' = 0 \Rightarrow x = 0, 1$. For $0 < x < 1$, $A' > 0$, and for $x > 1$, $A' < 0$. So $x = 1$ is a local maximum. Furthermore $x = 0$ is an endpoint where $A = 0$. Hence the maximum area for our triangle occurs when $x = 1$ and is $A = \frac{1}{2}e^{-3}$.

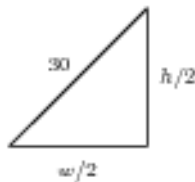


17. Let w and l be the width and length, respectively, of the rectangular area, A , you are to enclose.



$A = w \times l$ and $w + w + l = 100$ feet. So $l = 100 - 2w$ and $A = w(100 - 2w) = 100w - 2w^2$. $A' = 100 - 4w$ and $A' = 0 \Rightarrow w = 25$. $A'' = -4 < 0$ so $w = 25$ gives a local maximum. Our endpoints are when $w = 0$ or $l = 0$ and both of these cases yield $A = 0$. So our maximum area occurs when $w = 25$ (and $l = 50$) and is $A = 1250$.

18. Let S be the strength of the beam. Then $S = kwh^2$ for some constant k . Considering the triangle below,



we also have by Pythagoras' Theorem that $(\frac{w}{2})^2 + (\frac{h}{2})^2 = 30^2$ or $w^2 + h^2 = 3600$. So $h^2 = 3600 - w^2$ and $S = kw(3600 - w^2) = k(3600w - w^3)$. $S' = k(3600 - 3w^2)$ and $S' = 0 \Rightarrow w = \pm\sqrt{1200} = \pm 20\sqrt{3}$. Obviously $w = -20\sqrt{3}$ does not make sense for our problem. Also $S'' = -6w$ and when $w = 20\sqrt{3}$, $S'' < 0$, so we have a local maximum. When $w = 20\sqrt{3}$, $h^2 = 3600 - 1200 = 2400$ or $h = 20\sqrt{6}$ (- ignoring negative h for our problem). So the beam with maximum strength will have width $20\sqrt{3}$ cm and height $20\sqrt{6}$ cm.

19. Consider the rectangular region with sides x and y . Then $xy = 3000$ or $y = 3000/x$. Suppose the three sides with shrubs are y and y and x . And the side with fencing is x . Then the cost, C , is calculated as $C = 25(x+2y)+10x = 35x+50y = 35x+150000/x$. $C' = 35-150000/x^2$ and $C' = 0 \Rightarrow x \approx 65.5$ ft (- ignoring negative x for our problem). $C'' = 300000/x^3$ and when $x > 0$, $C'' > 0$, so $x \approx 65.5$ is a local minimum. Our interval is $x > 0$. When $x \rightarrow 0$ and $x \rightarrow \infty$, $C \rightarrow \infty$. So our local minimum is our global minimum and the minimum cost occurs when $x \approx 65.5$ and is approximately \$4583.

32. The distance from S to Town 1 is $\sqrt{1+x^2}$ and the distance from S to Town 2 is $\sqrt{(4-x)^2+4^2} = \sqrt{x^2-8x+32}$. So the total length of pipe, $f(x)$ say, is $f(x) = \sqrt{1+x^2} + \sqrt{x^2-8x+32}$. $f'(x) = \frac{2x}{2\sqrt{1+x^2}} + \frac{2x-8}{2\sqrt{x^2-8x+32}} = \frac{x\sqrt{x^2-8x+32}+(x-4)\sqrt{1+x^2}}{\sqrt{1+x^2}\sqrt{x^2-8x+32}}$. Now $f'(x) = 0 \Rightarrow x\sqrt{x^2-8x+32} + (x-4)\sqrt{1+x^2} = 0$. To solve this equation take $x\sqrt{x^2-8x+32} = -(x-4)\sqrt{1+x^2}$ and square both sides to give $x^2(x^2-8x+32) = (x^2-8x+16)(1+x^2)$ or $15x^2+8x-16 = (3x+4)(5x-4) = 0$. This yields the critical point $x = 4/5$. (Discard $x = -4/3$ since we are

only interested in x between 0 and 4, between the two towns.) Now $f'(0) < 0$ and $f'(1) > 0$ so $x = 4/5$ gives the location of S in order to minimize the length of pipe.

3. Section 4.8: 5, 6, 16, 20, 23, 27, 28, 30, 34.

Solution:

5. The motion is clockwise for $0 \leq t \leq 2\pi$.

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16. One possible answer is $x = 3 \cos(t)$, $y = -3 \sin(t)$, $0 \leq t \leq 2\pi$.

20. The slope of the line is $(3 - (-1))/(1 - 2) = -4$. The equation of the line with slope -4 through the point $(2, -1)$ is $y + 1 = -4(x - 2)$. So one possible parametrization is $x = t$, $y = -4t + 7$.

23. We have $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2-1}$. So when $t = 2$, the slope of the tangent is $4/11$. Also when $t = 2$, we have $x = 6$ and $y = 4$. Therefore the equation of the tangent line is $y - 4 = \frac{4}{11}(x - 6)$.

27. (a) With $t \geq 0$, $x \geq 2$ and $y \geq 4$. So we are restricted to the part of the line to the right of and above the point $(2, 4)$.

(b) When $t = 0$, $(x, y) = (2, 4)$, and when $t = -1$, $(x, y) = (-1, -3)$. So for $-1 \leq t \leq 0$ have the part of the line between these two points.

(c) $x < 0 \Rightarrow 2 + 3t < 0$ or $t < -2/3$.

28. (a) Eliminating t between $x = 2 + t$, $y = 4 + 3t$ gives $y = 3x - 2$. Similarly for eliminating t between $x = 1 - 2t$, $y = 1 - 6t$. Since both parametric equations give rise to the same equation in x and y , they both parametrize the same line.

(b) Slope = 3, y -intercept = -2 .

30. (a) C_1 has the origin as centre and radius 5. So $a = b = 0$, $k = 5$ or -5 .

(b) C_2 has $(0, 5)$ as centre and radius 5. So $a = 0$, $b = 5$, $k = 5$ or -5 .

(c) C_3 has $(10, -10)$ as centre. So $a = 10$ and $b = -10$. The radius is $\sqrt{10^2 + (-10)^2} = \sqrt{200}$ so $k = \sqrt{200}$ or $-\sqrt{200}$.

34. (a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4e^{2t}}{e^t} = 4e^t$.

(b) $y = 2e^{2t} = 2(e^t)^2 = 2x^2$.

(c) $\frac{dy}{dx} = 4x$. Note that since $x = e^t$, as expected this corresponds to our result in part (a).