The University of Sydney

MATH1901/06 Differential Calculus (Advanced)

Semester 1 **Assignment 1** 2009

This assignment is due by 4:00pm on **Tuesday, 24 March, 2009**. It should be posted in the glass-fronted collection boxes on the verandah of Carslaw Level 3. These boxes are at the end of the verandah closest to Eastern Avenue. (NOT the glass-fronted collection boxes near the pyramids on Carslaw Level 3, nor the open wooden pigeonholes.) Please do not post your assignment before the due date since the boxes are also used for the collection of assignments in other units. Your assignment must be stapled inside a manilla folder, and a cover sheet must be signed and attached.

This assignment is worth 5% of the assessment for this unit of study. It covers topics on complex numbers from the first two weeks of lectures.

1. [*7 marks*] Find all complex numbers z (in Cartesian form $x + iy$) that satisfy the following equations:

(a) [2 marks] $z^2 - (4+i)z + 5 - i = 0$;

(b) [3 marks] $az^2 + b\bar{z} + c = 0$, where a, b and c are positive real numbers with $b^2 - 4ac > 0$.

(c) [2 marks] $e^z = -2$. (You may assume at this stage that a complex exponential is defined by $e^{x+iy} = e^x \operatorname{cis} y$.)

2. [*5 marks*] Find all complex numbers z such that $\operatorname{Re}(z^3) < 0$ and show the solutions graphically in the complex plane. (This problem is best handled in polar coordinates.)

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3. [*8 marks*] A question on the exercise sheet in Tutorial Week 2 yielded the trigonometric identities,

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta,$$

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

This question takes you through two different ways to exactly evaluate $\sin(\pi/10)$ and $\sin(3\pi/10)$ using these identities. (The solutions, when they are released, will also describe a third method.)

(a) Because $\sin(\pi/2) = 1$, the second equation above implies that $y = \sin(\pi/10)$ is one of the roots of the quintic (degree 5) equation,

$$16y^5 - 20y^3 + 5y - 1 = 0.$$

(i). [2 marks] Use the $\sin 5\theta$ formula to explain why $y = \sin(\pi/10)$ and $y = -\sin(3\pi/10)$ are both roots of this quintic. Then check by direct substitution that $y = 1$ is also a root. (This identifies three of the five roots expected in the complex domain.)

(ii). [2 marks] Since $y = 1$ is known to be a root, we can divide out the factor $y-1$, leaving a quartic (degree 4) equation for y. Show that this quartic is, in fact, a perfect square of a quadratic expression. (Do not try to explain why this happens!) Solve the quadratic equation for y that you just found (using the quadratic formula) and identify its two roots with $\sin(\pi/10)$ and $-\sin(3\pi/10)$.

(b) Let $z = \sin(\pi/10) + i\cos(\pi/10)$. (This is the start of a second derivation of the exact values of $\sin(\pi/10)$ and $\sin(3\pi/10)$, so do not use the values that you found in part (a).)

(i). [1 mark] Show that $z = \text{cis}(2\pi/5)$ and that $z^5 = 1$.

(ii). [2 marks] Treat $z^5 - 1 = 0$ as a quintic equation for z. You may assume that its five roots are $z = \text{cis}(2k\pi/5)$ for $k = 0, \pm1, \pm2$. Divide out the simple factor $z - 1$ to obtain a quartic equation for z, and show that your quartic can be cast in the form,

$$\left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) + 1 = 0.$$

Obtain a quadratic equation for the variable $w = z + 1/z$ and find its two roots using the quadratic formula.

(iii). [1 mark] Starting with the four non-real roots of the original quintic equation $z^5 - 1 = 0$ in "cis" notation, substitute these into $w = z + 1/z$ and deduce that the two roots of the quadratic equation in part (b)(ii) must be $2\sin(\pi/10)$ and $-2\sin(3\pi/10)$.