The University of Sydney
MATH1901/06 Differential Calculus (Advanced)

Semester 1 **Assignment 1 solutions** 2009

Due Tuesday, 24 March, 2009.

This assignment is worth 5% of the assessment for this unit of study.

1. [*7 marks*] Find all complex numbers z (in Cartesian form $x+iy$) that satisfy the following equations:

(a) [2 marks] $z^2-(4+i)z+5-i = 0$;

(b) [3 marks] $az^2+b\bar{z}+c = 0$, where a, b and c are positive real numbers with $b^2-4ac>0$.

(c) [2 marks] $e^z=-2$. (You may assume at this stage that a complex exponential is defined by $e^{x+iy}=e^x\operatorname{cis} y$.)

Solution

(a) [2 marks] This is a quadratic equation for z with complex coefficients. We can get two values of z (that are not yet in the required form) by applying the quadratic formula with $a=1$, $b=-(4+i)$ and $c=5-i$. The result is

$$\begin{aligned} z &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{4+i\pm\sqrt{(4+i)^2-4(5-i)}}{2} \\ &= \frac{4+i\pm\sqrt{-5+12i}}{2}. \end{aligned}$$

The same result can be obtained by completing the square in the original quadratic equation and arriving at

$$\left(z-\tfrac{1}{2}(4+i)\right)^2 = \tfrac{1}{4}(-5+12i).$$

To complete the solution, we need at least one of the values of $\sqrt{-5+12i}$ in Cartesian form. These can be found quickly with a bit of trial and error, or by looking up similar cases in the solutions of tutorial exercises. So students do not need to show detailed working here. The method (which would be needed in more complicated cases where trial and error is not feasable) goes as follows.

Let the principal value of the required square root be $p + iq$ with $p > 0$. Then

$$\begin{aligned} (p+iq)^2 &= -5+12i, \\ p^2 - q^2 + 2ipq &= -5+12i, \\ p^2 - q^2 = -5 \quad &\text{and} \quad 2pq = 12. \end{aligned}$$

The last equation gives $q = 6/p$ and then the preceding equation becomes $p^2 - 36/p^2 = -5$, which can be rearranged to

$$p^4 + 5p^2 - 36 = (p^2+9)(p^2-4) = 0.$$

Since p is real, $p^2 + 9$ cannot vanish. So $p^2 = 4$ and then $p = 2$ since we asked for the principal value ($p > 0$). Then $q = 6/p = 3$. So the required principal square root is

$$\sqrt{-5+12i} = 2+3i.$$

Returning now to the quadratic formula, we find that the two roots of the given quadratic equation for z are

$$\begin{aligned} z &= \frac{4+i \pm \sqrt{-5+12i}}{2} \\ &= \frac{4+i \pm (2+3i)}{2} \\ &= \boxed{3+2i} \quad \text{and} \quad \boxed{1-i.} \end{aligned}$$

You should check directly that these roots are correct by substituting them into the quadratic equation.

(b) [3 marks] Because of the appearance of $\bar{z}$, this equation is not a quadratic equation for z. Write $z = x + iy$ and $\bar{z} = x - iy$ and substitute into the given equation:

$$\begin{aligned} 0 &= az^2 + b\bar{z} + c \\ &= a(x^2 - y^2 + 2ixy) + b(x - iy) + c \\ &= (ax^2 - ay^2 + bx + c) + i(2axy - by). \end{aligned}$$

Since the coefficients a, b and c are real, we can separate the given equation for z into two equations for the real variables x and y. Thus

$$ax^2 - ay^2 + bx + c = 0 \quad \text{and} \quad y(2ax - b) = 0.$$

The second equation factorizes and gives us two cases to consider.

Case 1: $y = 0$. Then z is real, $z = x$, and $ax^2 + bx + c = 0$. Since we are given that $b^2 - 4ac > 0$, we know that this equation has two distinct real roots. So two values of z in Case 1 have been found, namely,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Case 2: $x = b/(2a)$. Substituting into the above equation, $ax^2 - ay^2 + bx + c = 0$ gives

$$\frac{b^2}{4a} - ay^2 + \frac{b^2}{2a} + c = 0,$$

$$ay^2 = \frac{3b^2}{4a} + c,$$

$$y^2 = \frac{3b^2 + 4ac}{4a^2}.$$

We will get two distinct real values of y whenever $3b^2 + 4ac > 0$. Since the coefficients were stated to be positive, this condition is satisfied. So we have arrived at two more values of z in Case 2, namely,

$$z = \frac{b \pm i\sqrt{3b^2 + 4ac}}{2a}.$$

So the given equation has four roots, two real and two nonreal under the stated conditions. Collecting results, we have

$$\boxed{z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad \frac{b \pm i\sqrt{3b^2 + 4ac}}{2a}.}$$

(c) [2 marks] This question asks us to find all the complex logarithms of -2. Start with

$$\begin{aligned} -2 = e^z &= e^{x+iy} \\ &= e^x\, e^{iy} \\ &= e^x \operatorname{cis} y \\ &= e^x(\cos y + i \sin y). \end{aligned}$$

As a general rule when taking logarithms, it is better to take the modulus of both sides first, rather than the real and imaginary parts. Since $\operatorname{cis} y$ always has modulus 1, we get $e^x = 2$ straight away. Since x is real, this has the unique solution $x = \ln 2$. Then $\cos y = -1$ and $\sin y = 0$. We can read off $y = (2n+1)\pi$, where n runs through the integers. So the required values of z satisfying $e^z = -2$ are

$$\boxed{z = \ln 2 + (2n+1)\pi i, \qquad n \in \mathbf{Z}.}$$

2. [*5 marks*] Find all complex numbers z such that $\text{Re}(z^3) < 0$ and show the solutions graphically in the complex plane. (This problem is best handled in polar coordinates.)

Solution

[3 marks for any correct description of the domain, 2 marks for the diagram.] The quickest way to find the required domain is to use the polar form,

$$z = r \operatorname{cis} \theta = r(\cos\theta + i \sin\theta).$$

Then, according to De Moivre's theorem,

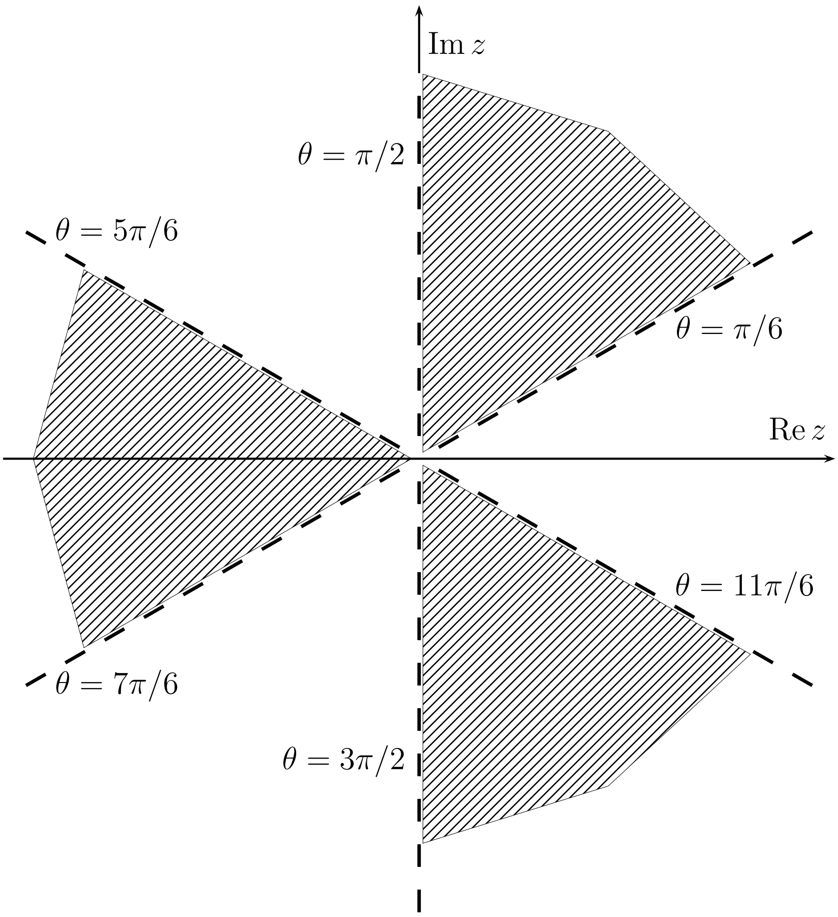
$$\begin{aligned} z^3 &= r^3 \operatorname{cis} 3\theta = r(\cos 3\theta + i \sin 3\theta), \\ \text{Re}(z^3) &= r \cos 3\theta, \\ \text{Im}(z^3) &= r \sin 3\theta. \end{aligned}$$

So $\text{Re}(z^3) < 0$ whenever $\cos 3\theta < 0$. On the θ-interval $[0, 2\pi)$, the cosine is negative on three open subintervals:

$$\frac{\pi}{6} < \theta < \frac{\pi}{2}, \qquad \frac{5\pi}{6} < \theta < \frac{7\pi}{6}, \qquad \frac{3\pi}{2} < \theta < \frac{11\pi}{6}.$$

[Students can express these θ-intervals in other ways. For example, in terms of principal arguments, the third interval is $-\pi/2 < \theta < -\pi/6$ while the second interval needs to be split into $5\pi/6 < \theta \leq \pi$ and $-\pi < \theta < -5\pi/6$. Students may also solve this problem in Cartesian coordinates: $\text{Re}(z^3) = x(x^2 - 3y^2) < 0$ implies $|y| > x/\sqrt{3}$ when $x > 0$ and $|y| < |x|/\sqrt{3}$ when $x < 0$.]

In the argand diagram, draw six rays from the origin at angles $\theta = \pi/6$, $\pi/2$, $5\pi/6$, $7\pi/6$, $3\pi/2$ and $11\pi/6$, and draw them with dotted or dashed lines because they are boundaries that are not included. These rays divide the complex z-plane into six sectors. Shade the three sectors above where $\cos 3\theta < 0$. (In the diagram on the next page, the shaded sectors run out to complex infinity, so disregard the apparent outer boundaries of the shaded areas.)



3. [*8 marks*] A question on the exercise sheet in Tutorial Week 2 yielded the trigonometric identities,

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta,$$
$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

This question takes you through two different ways to exactly evaluate $\sin(\pi/10)$ and $\sin(3\pi/10)$ using these identities. (The solutions, when they are released, will also describe a third method.)

(a) Because $\sin(\pi/2) = 1$, the second equation above implies that $y = \sin(\pi/10)$ is one of the roots of the quintic (degree 5) equation,

$$16y^5 - 20y^3 + 5y - 1 = 0.$$

(i). [2 marks] Use the $\sin 5\theta$ formula to explain why $y = \sin(\pi/10)$ and $y = -\sin(3\pi/10)$ are both roots of this quintic. Then check by direct substitution that $y = 1$ is also a root. (This identifies three of the five roots expected in the complex domain.)

(ii). [2 marks] Since $y = 1$ is known to be a root, we can divide out the factor $y-1$, leaving a quartic (degree 4) equation for y. Show that this quartic is, in fact, a perfect square of a quadratic expression. (Do not try to explain why this happens!) Solve the quadratic equation for y that you just found (using the quadratic formula) and identify its two roots with $\sin(\pi/10)$ and $-\sin(3\pi/10)$.

(b) Let $z = \sin(\pi/10) + i\cos(\pi/10)$. (This is the start of a second derivation of the exact values of $\sin(\pi/10)$ and $\sin(3\pi/10)$, so do not use any results that you found in part (a).)

(i). [1 mark] Show that $z = \text{cis}(2\pi/5)$ and that $z^5 = 1$.

(ii). [2 marks] Treat $z^5 - 1 = 0$ as a quintic equation for z. You may assume that its five roots are $z = \text{cis}(2k\pi/5)$ for $k = 0, \pm1, \pm2$. Divide out the simple factor $z - 1$ to obtain a quartic equation for z, and show that your quartic can be cast in the form,

$$\left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) + 1 \;=\; 0.$$

Obtain a quadratic equation for the variable $w = z + 1/z$ and find its two roots using the quadratic formula.

(iii). [1 mark] Starting with the four non-real roots of the original quintic equation $z^5 - 1 = 0$ in "cis" notation, substitute these into $w = z + 1/z$ and deduce that the two roots of the quadratic equation in part (b)(ii) must be $2\sin(\pi/10)$ and $-2\sin(3\pi/10)$.

Solution

(a) (i). [2 marks] Let $y = \sin(\pi/10)$. The $\sin 5\theta$ formula gives

$$16y^5 - 20y^3 + 5y \;=\; \sin(5\pi/10) \;=\; \sin(\pi/2) \;=\; 1.$$

Similarly let $y = -\sin(3\pi/10) = \sin(-3\pi/10)$. Then

$$16y^5 - 20y^3 + 5y \;=\; \sin(-15\pi/10) \;=\; \sin(-3\pi/2) \;=\; 1.$$

So $\sin(\pi/10)$ and $-\sin(3\pi/10)$ are both roots of the quintic equation,

$$16y^5 - 20y^3 + 5y - 1 \;=\; 0.$$

We know that every quintic equation has exactly five roots in the complex domain, counting double roots as two roots, and so on. We have just found two of the five roots. A third root $y = 1$ can be verified directly by substitution:

$$16 \cdot 1^5 - 20 \cdot 1^3 + 5 \cdot 1 - 1 \;=\; 16 - 20 + 5 - 1 \;=\; 0.$$

Remarks. The root $y = 1$ can also be explained by putting $\theta = \pi/2$ in the $\sin 5\theta$ formula. If we continue using the $\sin 5\theta$ formula in this fashion, the next two roots will turn out to be $\sin(9\pi/10)$ and $\sin(13\pi/10)$, but these are just copies of the roots $\sin(\pi/10)$ and $-\sin(3\pi/10)$ that we already know. This suggests that the first two roots are actually double roots, which will be proved part (ii).

(ii). [2 marks] The root $y = 1$ of the quintic in part (i) implies that $y - 1$ is a divisor of the quintic polynomial. Dividing out the factor $y - 1$ gives

$$16y^5 - 20y^3 + 5y - 1 \;=\; (y-1)(16y^4 + 16y^3 - 4y^2 - 4y + 1).$$

[There are several ways to do this, and students do not need to explain their method in detail.] The question asked you to show the quartic factor is actually a perfect square of a quadratic expression. This can be found by trial and error, or by undetermined coefficients. For example, let

$$16y^4 + 16y^3 - 4y^2 - 4y + 1 \;=\; (ay^2 + by + c)^2,$$

with $a > 0$. Then, $a^2 = 16$ implies $a = 4$. Next, $2ab = 16$ implies $b = 2$ and $2bc = -4$ implies $c = -1$. Then one should multiply out the right-hand side to confirm that the identity,

$$16y^4 + 16y^3 - 4y^2 - 4y + 1 \;=\; (4y^2 + 2y - 1)^2,$$

is indeed correct. We now know that $\sin(\pi/10)$ and $-\sin(3\pi/10)$ are the two roots of the quadratic equation,

$$4y^2 + 2y - 1 \;=\; 0,$$

and are double roots of the original quintic equation. (So all five roots are now accounted for.) The quadratic formula give the two roots,

$$y \;=\; \frac{-1 \pm \sqrt{5}}{4}.$$

Observe that one root is positive and one is negative. This implies the exact evaluations,

$$\boxed{\sin(\pi/10) \;=\; \frac{\sqrt{5}-1}{4}, \qquad \sin(3\pi/10) \;=\; \frac{\sqrt{5}+1}{4}.}$$

(b) (i). [1 mark] Part (a) stayed in the real domain throughout. This part shows a quicker method using complex variables. We are given the definition $z = \sin(\pi/10) + i\cos(\pi/10)$. The identities $\sin\theta = \cos(\pi/2 - \theta)$ and $\cos\theta = \sin(\pi/2 - \theta)$ imply that

$$\sin(\pi/10) \;=\; \cos(2\pi/5), \qquad \cos(\pi/10) \;=\; \sin(2\pi/5).$$

The given value of z becomes

$$z = \cos(2\pi/5) + i\sin(2\pi/5) = \operatorname{cis}(2\pi/5) = e^{2\pi i/5}.$$

De Moivre's theorem immediately gives

$$\boxed{z^5 = \operatorname{cis}(2\pi) = 1.}$$

Thus the given z is one of the fifth roots of unity.

(ii). [2 marks] This part asks you to treat the equation,

$$z^5 - 1 = 0,$$

as a quintic equation for the variable z. Be aware that this is a different usage of the symbol z. The particular z in part (i) is one of the five roots of this quintic. The full set of roots is $z = \operatorname{cis}(2k\pi/5)$ for $k = 0, \pm1, \pm2$ (or, equivalently, $k = 0, 1, 2, 3, 4$).

The quintic $z^5 - 1 = 0$ has the obvious root $z = 1$ (the principal fifth root of unity) and so $z - 1$ must be a factor of the left-hand side. Dividing it out gives

$$z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1).$$

The four non-real fifth roots of unity are the roots of the quartic equation,

$$z^4 + z^3 + z^2 + z + 1 = 0.$$

Dividing by z^2 and rearranging gives

$$\left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) + 1 = 0.$$

This is a useful form for the quartic. The question asks you to apply the change of variable, $w = z + 1/z$. Then $w^2 = z^2 + 1/z^2 + 2$ and the quartic becomes

$$(w^2 - 2) + w + 1 = 0,$$

which can be rearranged to the quadratic equation,

$$\boxed{w^2 + w - 1 = 0.}$$

The quadratic formula gives the two roots,

$$\boxed{w = \frac{-1 \pm \sqrt{5}}{2}.}$$

(iii). [1 mark] Begin with the four non-real roots $z = \operatorname{cis}(2k\pi/5)$, $k = \pm1, \pm2$, of the quintic equation $z^5 - 1 = 0$. The identity $\operatorname{cis}(-\theta) = 1/\operatorname{cis}\theta$ provides a short cut. The values of w corresponding to the cases $k = 1$ and $k = 2$ are

$$w \;=\; z + \frac{1}{z} \;=\; \text{cis}(2\pi/5) + \text{cis}(-2\pi/5) \;=\; 2\cos(2\pi/5) \;=\; 2\sin(\pi/10),$$

$$w \;=\; z + \frac{1}{z} \;=\; \text{cis}(4\pi/5) + \text{cis}(-4\pi/5) \;=\; 2\cos(4\pi/5) \;=\; -\,2\sin(3\pi/10).$$

The cases $k = -1$ and $k = -2$ give the same two values of w. As in part (a), the quadratic equation for w has two real roots, one positive and one negative. Hence we arrive again at the exact evaluations,

$$\boxed{\sin(\pi/10) \;=\; \frac{\sqrt{5}\,-1}{4}\,, \qquad \sin(3\pi/10) \;=\; \frac{\sqrt{5}\,+1}{4}\,.}$$

(Of course, we could finish calculating the fifth roots of unity by inverting $w = z + 1/z$, but we have already reached our desired goal, and so we can stop here.)

Concluding remarks. The previous two methods evaluated $\sin(\pi/10)$ and $\sin(3\pi/10)$ by reducing a quintic equation to a quadratic equation. Observant students might wonder why we overlooked a much more obvious quadratic implied by the $\cos 5\theta$ formula. The fact that $\cos(\pi/2) = 0$ implies that $x = \cos(\pi/10)$ is one of the four roots of the even quartic equation,

$$16x^4 - 20x^2 + 5 \;=\; 0,$$

which is a quadratic equation for x^2. The full set of roots is $x = \pm\cos(\pi/10)$ and $\pm\cos(3\pi/10)$. Applying the quadratic formula gives

$$x^2 \;=\; \frac{20 \pm \sqrt{80}}{32} \;=\; \frac{5 \pm \sqrt{5}}{8}\,.$$

The next square root does not simplify. The best we can do is

$$\cos(\pi/10) \;=\; \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}\,, \qquad \cos(3\pi/10) \;=\; \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}\,.$$

(A nested square root $\sqrt{a + b\sqrt{c}}$ simplifies only when $a^2 - b^2 c$ is a perfect square.) The corresponding sines can be calculated from the Pythagoras identity, $\sin^2\theta + \cos^2\theta = 1$. This gives

$$\sin(\pi/10) \;=\; \frac{\sqrt{3-\sqrt{5}}}{2\sqrt{2}}\,, \qquad \sin(3\pi/10) \;=\; \frac{\sqrt{3+\sqrt{5}}}{2\sqrt{2}}\,.$$

These nested square roots do simplify. One can check by squaring that

$$\sin(\pi/10) \;=\; \frac{\sqrt{5}\,-1}{4}\,, \qquad \sin(3\pi/10) \;=\; \frac{\sqrt{5}\,+1}{4}\,.$$

So the "obvious" quadratic was not the best way to approach this problem. The methods in parts (a) and (b) are somewhat more elegant.