The University of Sydney

MATH1901/06 Differential Calculus (Advanced)

Semester 1 **Assignment 2 solutions** 2009

Due Tuesday, 26 May, 2009.

This assignment is worth 5% of the assessment for this unit of study.

1. [*9 marks, 3 for each part*] Calculate the following limits or prove that they do not exist. (Allow $+\infty$ and $-\infty$ as values that a limit can take.)

(a) $\displaystyle\lim_{x\to\infty} \sinh x \sinh(e^{-x})$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{(x^2+y^2)^2}$.

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4}{y+x^2}$, where the limit is taken along all possible paths that never intersect the parabola $y=-x^2$ except at their common endpoint $(0,0)$.

Solution

(a) [*3 marks.*] When x is large, $\sinh x$, which equals $(e^x - e^{-x})/2$, grows exponentially like $(e^x)/2$. So the given limit will be one half of the limit of $\sinh(e^{-x})/(e^{-x})$ as $x\to\infty$, which is a standard $0/0$-type limit that can be handled with l'Hôpital's rule. One way to set this out is as follows:

$$\begin{aligned}
\lim_{x\to\infty} \sinh x \sinh(e^{-x}) &= \lim_{x\to\infty} \frac{e^x - e^{-x}}{2} \sinh(e^{-x}) \\
&= \lim_{x\to\infty} \frac{1-e^{-2x}}{2} \frac{\sinh(e^{-x})}{e^{-x}} \\
&= \frac{1}{2} \lim_{x\to\infty} \frac{\sinh(e^{-x})}{e^{-x}} && \text{(0/0-type limit)} \\
&= \frac{1}{2} \lim_{x\to\infty} \frac{-e^{-x}\cosh(e^{-x})}{-e^{-x}} && \text{(l'Hôpital's rule)} \\
&= \frac{1}{2} \lim_{x\to\infty} \cosh(e^{-x}) \\
&= \frac{1}{2}\cosh(0) \\
&= \boxed{\frac{1}{2}.}
\end{aligned}$$

Remark. L'Hôpital's rule can be bypassed by observing that

$$\lim_{x\to\infty} \frac{\sinh(e^{-x})}{e^{-x}} = \lim_{y\to 0^+} \frac{\sinh y}{y} = 1,$$

because $e^{-x} \to 0^+$ as $x \to \infty$. The latter limit is a standard limit (and is two-sided). It is obvious from l'Hôpital's rule or from the limit that defines the derivative $\sinh' 0 = \cosh 0 = 1$. Alternatively, it can be read off the Taylor polynomial $T_1(y) = y$ for $\sinh y$ about $y = 0$. (The full Taylor series is $\sinh y = y + y^3/3! + y^5/5! + y^7/7! + \ldots = -\,i\sin(iy)$.)

(b) [*3 marks.*] In the case of the function $f(x, y) = x^4y^2/(x^2 + y^2)^2$ for $(x, y) \neq (0, 0)$, we would expect a limit of zero at the origin because the numerator tends to zero more quickly than the denominator as $(x, y) \to (0, 0)$. We need to show that the limit is zero along ALL paths that end at the origin. This is best done in polar coordinates, especially when $x^2 + y^2$ appears in the expression for the function. Let

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

where r (the polar radial coordinate) is positive and θ (the polar angular coordinate) varies continuously along any particular path. (So θ is not necessarily confined to an interval such as $[0, 2\pi)$ or $(-\pi, \pi]$.) In polar coordinates,

$$\begin{aligned} f(x,y) &= \frac{x^4y^2}{(x^2+y^2)^2} && ((x,y)\neq(0,0)) \\ &= \frac{r^6\cos^4\theta\sin^2\theta}{r^4} && (r>0) \\ &= r^2\cos^4\theta\sin^2\theta. \end{aligned}$$

So we have the inequality,

$$0 \le |f(x, y)| \le r^2,$$

where the absolute value signs are optional (because only even powers occur). Now r and hence r^2 must tend to zero along every path that ends at the origin. So, by the squeeze lemma,

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^4y^2}{(x^2+y^2)^2} = \lim_{r\to 0} r^2 = 0.}$$

(c) [*3 marks.*] **Discussion**. In the case of the function $f(x, y) = x^4/(y + x^2)$ for $y \neq -x^2$, we should not expect a limit to exist as $(x, y) \to (0, 0)$ because suitably chosen paths that approach the origin close to the forbidden parabola $y = -x^2$ will give different limits or infinite limits. (An infinite limit would be allowed if it was the same on every path of approach, as in the case of

$1/(x^2+y^2)$.) We aim to exhibit two paths along which $f(x,y)$ tends to different limits. That will prove that $f(x,y)$ has no limit as $(x,y) \to (0,0)$.

First observe that the limit along any straight path to the origin is zero. On the x-axis, $f(x,0) = 0$, which tends to zero as $x \to 0$. On the y-axis, $f(0,y) = 0$, which tends to zero as $y \to 0$. On the line $y = mx$, $m \neq 0$, $f(x,mx) = x^3/(m+x)$, which tends to zero as $x \to 0$. (Do not be concerned if the latter lines cross the forbidden parabola $y = -x^2$ in the lower half-plane, as only the last part of the path near the origin is of any relevance to the limit. Such paths can be bent away from the parabola, if desired.)

Since we are aiming to prove that the limit of $f(x,y)$ as $(x,y) \to (0,0)$ does not exist, we can only use ONE of the paths in the previous paragraph. An obvious choice, for example, is the positive x-axis, along which $f(x,y)$ tends to zero.

To complete the proof, we need to find one path along which the limit is NOT zero (or is infinite, or does not exist). Such a path must be curved, and be close to the forbidden parabola $y = -x^2$ in some sense. The simplest such paths are those that are tangent to the parabola at the origin in a suitable sense.

Consider the family of paths $y = kx^n - x^2$ for $n > 2$. These are tangent to the parabola $y = -x^2$ at $x = 0$ and do not cross or touch it anywhere else. The case $n = 3$ is no good because we again arrive at the limit zero. However, the case $n = 4$ gives the path $y = kx^4 - x^2$, along which

$$f(x,y) \;=\; f(x,\, kx^4 - x^2) \;=\; \frac{x^4}{kx^4} \;=\; \frac{1}{k}\,,$$

which tends to $1/k$ (not zero) as $x \to 0$. So the paths of this family are among the **level curves** for the function $f(x,y) = x^4/(y+x^2)$, and they all approach the origin at different heights. We only need ONE of them, say, $y = x^4 - x^2$, along which the limit is 1. (Alternatively, we could pick two of these curved paths and disregard the straight paths discussed earlier.) So we have at least TWO paths giving different limits, and that proves that $f(x,y)$, as a function of two variables, does not tend to a limit as $(x,y) \to (0,0)$.

Alternatively, one could consider the family of paths $y = kx^5 - x^2$ or $y = kx^6 - x^2$. In the first case, we have one-sided limits of $+\infty$ and $-\infty$, which are different to each other, and we could stop right there. In the second case, we have two-sided limits of $+\infty$ (if k is positive) or $-\infty$ (if k is negative), and these are different to the finite limits found earlier and to each other. To conclude, we have found paths along which $f(x,y)$ tends to any finite real number or to $+\infty$ or $-\infty$. We only need to pick out two particular paths on which the limits are different to prove that $f(x,y)$ does not tend to a limit as $(x,y) \to (0,0)$.

Short answer. According to the above preamble, we can give a quick answer to the question as follows:

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \qquad \text{(path to origin along positive } x\text{-axis)},$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1 \qquad \text{(path to origin along curve } y = x^4 - x^2).$$

Having found different limits along different paths, we conclude that

$$\boxed{\lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist.}}$$

Any two paths giving different limits (or a single path giving no limit) will do the job.

Exercise (not examinable). Find a single path, ending at the origin and not crossing the forbidden parabola, along which $f(x,y)$ is bounded but does not tend to a limit.

2. [*6 marks, 3 for each part*] Find the absolute maximum and minimum values taken by the following functions on the interval $[2,3]$:

(a) $f(x) = (1 - 1/x)^{-x}$, (b) $g(x) = x^{1/x}$.

Solution

(a) [*3 marks.*] The function $f(x) = (1 - 1/x)^{-x}$ is similar to a case that appeared on Sample Quiz 2 and can be handled by the same method. Based on that experience, we should expect that $f(x)$ is monotonic for $x > 1$, and so its extrema on the interval $[2,3]$ will occur at the endpoints. Since $f(2) = 4$ and $f(3) = 27/8$, we aim to prove that $f(x)$ is decreasing on $[2,3]$ (or, more generally, on $(1,\infty)$).

Let $g(x) = \ln f(x) = -x \ln(1 - 1/x)$. Then $f(x)$ is decreasing whenever $g(x)$ is decreasing and vice-versa. Take a derivative using the product and chain rules:

$$\begin{aligned} g'(x) &= -\ln\left(1 - \frac{1}{x}\right) - x\,\frac{1/x^2}{1 - 1/x} \\ &= -\ln\left(1 - \frac{1}{x}\right) - \frac{1}{x-1}, \end{aligned}$$

for $x > 1$. (We could have used the identity $\ln(1 - 1/x) = \ln(x-1) - \ln x$.) Observe that $g(x) \to 0$ as $x \to \infty$. Since it is probably not yet obvious whether $g'(x)$ is positive or negative, take another derivative:

$$g''(x) = -\frac{1/x^2}{1 - 1/x} + \frac{1}{(x-1)^2} = \frac{1}{x(x-1)^2}.$$

So $g''(x)$ is positive for all $x > 1$, which implies that $g'(x)$ is increasing. Since $g'(x)$ increases towards zero as $x \to \infty$, we now know that $g'(x)$ is negative for $x > 1$. This completes the proof that $g(x)$ and $f(x)$ are monotonic decreasing for $x > 1$. The answer to the question is, on the interval $[2, 3]$,

$$f(x) \text{ takes its absolute maximum at } x = 2, \text{ value } \boxed{f(2) = 4,}$$

$$f(x) \text{ takes its absolute minimum at } x = 3, \text{ value } \boxed{f(3) = 27/8.}$$

(b) [*3 marks.*] The function $g(x) = x^{1/x}$ (the "xth root of x") is easier to handle. Its domain is $x \geq 0$ if we define $g(0) = 0$. Let

$$h(x) \;=\; \ln g(x) \;=\; \frac{\ln x}{x},$$

$x > 0$. Then the quotient rule gives

$$h'(x) \;=\; \frac{g'(x)}{g(x)} \;=\; \frac{1 - \ln x}{x^2}, \qquad g'(x) \;=\; x^{1/x}\,\frac{1 - \ln x}{x^2}.$$

We see that $g'(x) > 0$ and $g(x)$ is increasing on the interval $(0, e)$ and $g'(x) < 0$ and $g(x)$ is decreasing on the interval (e, ∞). The graph of $y = g(x)$ has a horizontal tangent at $x = e$. So $g(x)$ has an absolute maximum at $x = e$ on any interval that contains e. The given interval $[2, 3]$ contains e as an interior point (recall $e = 2.71828\ldots$). So the absolute maximum value taken by $g(x) = x^{1/x}$ on the interval $[2, 3]$ is

$$\boxed{g(e) \;=\; e^{1/e}} \;=\; 1.444667861\ldots.$$

On the other hand, the absolute minimum must occur at one of the endpoints (since there are no critical points in the interior besides e). A calculator is helpful here. We find

$$g(2) \;=\; 2^{1/2} \;=\; 1.414213562\ldots, \qquad g(3) \;=\; 3^{1/3} \;=\; 1.442249570\ldots.$$

So $g(2) < g(3)$. Without a calculator, one can see that $g(2)^6 = 2^3 = 8$ and $g(3)^6 = 3^2 = 9$, and that also implies that $g(2) < g(3)$. Either way, we have found that the absolute minimum occurs at the left endpoint $x = 2$ with value,

$$\boxed{g(2) \;=\; \sqrt{2}} \;=\; 1.414213562\ldots.$$

3. [*5 marks*] Let $G(x) = \frac{1}{2}\big\{\cos(x^{1/4}) + \cosh(x^{1/4})\big\}$ for $x \geq 0$. Use suitable Taylor polynomials to find the right derivatives $G'_+(0)$ and $G''_+(0)$ at $x = 0$.

Solution

[*5 marks, minimum of 3 marks if the student understands the method.*] This problem is similar to a problem on Quiz 2 that was not handled well by the class. A major hint for this assignment question was given in the Friday lecture after the quiz showing how the quiz question should have been done. A solution to the quiz question itself is appended after this assignment solution.

We are given

$$G(x) \;=\; \tfrac{1}{2}\bigl\{\cos(x^{1/4}) + \cosh(x^{1/4})\bigr\}, \qquad x \geq 0.$$

We want to use suitable Taylor polynomials for $\cos x$ and $\cosh x$ about $x = 0$. Before deciding on the order of these Taylor polynomials, it is handy to have the full Taylor series about $x = 0$,

$$\cos x \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\,,$$
$$\cosh x \;=\; 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\,,$$

which converge to their respective functions for all real x. (They also converge for all complex x as well, which is why we are justified in saying $\cosh x = \cos(ix)$ and $\cos x = \cosh(ix)$ for all $x \in \mathbf{C}$.) Adding gives

$$\tfrac{1}{2}\bigl\{\cos x + \cosh x\bigr\} \;=\; 1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \frac{x^{12}}{12!} + \ldots\,,$$

valid for all real x. For $x \geq 0$, we may replace x by $x^{1/4}$ and get

$$G(x) \;=\; 1 + \frac{x}{4!} + \frac{x^2}{8!} + \frac{x^3}{12!} + \ldots\,.$$

This infinite series (a power series in x) provides an extension of $G(x)$ to negative real x and to all complex x, but we do not need this extension here. If we keep the original domain $[0, \infty)$, then we have a one-sided power series for $G(x)$.

An important theorem in the lecture notes (page 88) allows us to generate new Taylor polynomials from old. In particular, it allows us to state that the quadratic polynomial,

$$T_2(x) \;=\; 1 + \frac{x}{4!} + \frac{x^2}{8!}\,,$$

is indeed the (one-sided) Taylor polynomial of order two for $G(x)$ about $x = 0$. In other words,

$$T_2(x) \;=\; G(0) + G'(0)x + \frac{G''(0)}{2!}x^2,$$

where the derivatives are understood to be right derivatives (with optional plus subscript as in $G'_+(0)$ and $G''_+(0)$). Hence, we can read off the right derivatives,

$$\boxed{G'(0) = \frac{1}{4!} = \frac{1}{24}\,,} \qquad \boxed{G''(0) = \frac{2!}{8!} = \frac{1}{20160}\,.}$$

More generally, $G^{(n)}(0) = n!/(4n)!$. These are two-sided derivatives if we extend $G(x)$ to negative x using its power series above. (If we had begun with Taylor polynomials for $\cos x$ and $\cosh x$, we would have needed them to order eight.)

Exercise (not examinable). Express $G(-x)$, $x > 0$, in terms of trigonometric and hyperbolic functions of real variables.

Remark. You should memorize the standard Taylor series for e^x, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, $\ln(1 + x)$ and the binomial series for $(1 + x)^\alpha$, which includes the case of the geometric series for $(1+x)^{-1}$, and also be able to put bounds on the remainder terms when the series are truncated to form Taylor polynomials. (The inverse tangent series is also standard, but you will not need to memorize it until MATH1903.) You do not want to have to derive the standard series from scratch each time you want to use them. Taylor polynomials of low degree for many compound expressions can be built from these standard cases by elementary operations (e.g., multiplication, long division, replacing x by x^2, and so on). (Check out Questions 5–7 on Sample Quiz 2 for more examples of new Taylor polynomials from old.)

Appendix: Solution to Quiz 2 Question 4. The four versions of the quiz question were all minor variants of the same question, namely, to calculate $f'(0)$ when

$$f(x) \;=\; \sinh(x^{1/3}) - \sin(x^{1/3}).$$

To answer the question, it is enough to know in advance that $\sinh x$ and $\sin x$ have the respective Taylor polynomials $x + x^3/3!$ and $x - x^3/3!$ of order three. Then $T_1(x)$ for $f(x)$ cancels to $x/3$, from which we read off $f'(0) = 1/3$.

Because of minor variations in the four quiz versions, only the pink sheet had the answer $f'(0) = 1/3$ to Question 4. The other three answers were $2/3$, -1 and $8/3$. You should revisit this question to confirm your particular answer.

Exercise. Find $T_2(x)$ for the above $f(x)$ and deduce the value of $f''(0)$. Show also that the graph of $y = f''(x)$ has a vertical tangent at $x = 0$, and so the Taylor polynomials for $f(x)$ about $x = 0$ terminate at order two.