Semester 1	**Tutorial Solutions Week 2**	2009

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Express the following in the form $x + iy$ (*Cartesian* or *standard* form):

(a) $(2 + 3i) + (4 - 5i)$;

(b) $(1 + i)(1 - i)$;

(c) $(2 + 3i) - (4 - 5i)$;

(d) $\dfrac{1+i}{1-i}$;

(e) $\dfrac{1+2i}{3-4i}$;

(f) $(1 + i)^2$;

(g) i^9;

(h) $i^{123} - 4i^8 - 4i$.

Questions for the Tutorial

2. Solve the following equations in $\mathbb{C}$:

(a) $z^2 + 3z + 2 = 0$

(b) $z^2 + z + 1 = 0$

(c) $z^2 + 2\overline{z} + 1 = 0$

(d) $z^4 = 16$

Solution

(a) $z^2 + 3z + 2 = (z + 2)(z + 1) = 0 \iff z = -2$ or $z = -1$.

(b) The quadratic formula is perfectly valid when solving quadratic equations in $\mathbb{C}$ (after all, it just comes from completing the square). So $z^2 + z + 1 = 0 \iff z = \dfrac{-1 \pm \sqrt{-3}}{2}$. The expression $\pm\sqrt{-3}$ means either of the two complex square roots of -3, namely $\pm i\sqrt{3}$. The solutions are therefore $z = \dfrac{-1 \pm i\sqrt{3}}{2}$.

(c) Note that we cannot use the quadratic formula here, because the middle term is $\overline{z}$, not z. Instead, we set $z = a + bi$ for a, b real. Then

$$z^2 + 2\overline{z} + 1 = (a^2 - b^2) + 2abi + 2(a - bi) + 1 = (a^2 - b^2 + 2a + 1) + (2ab - 2b)i.$$

This is zero if and only if both the real and imaginary parts are zero, i.e. $a^2-b^2+2a+1 = 0$ and $2ab-2b = 0$. The second equation can also be written as $2(a-1)b = 0$, which gives two cases: either $a = 1$ or $b = 0$. If $a = 1$, the first equation becomes $1-b^2+2+1 = 0$, which has solutions $b = \pm 2$. If $b = 0$, then the first equation becomes $a^2 + 2a + 1 = 0$, which has solution $a = -1$. So the solutions to the original equation are $z = -1$ and $z = 1 \pm 2i$.

(d) $z^4 = 16 \iff z^2 = \pm 4 \iff z = \pm 2, \pm 2i$.

3. (a) Find all solutions of the equation $z^2 + 3 + 4i = 0$ by setting $z = a + bi$ for some real numbers a and b.

(b) Solve $z^2 + z + 1 + i = 0$. (*Hint*: Use your solution to the previous part.)

Solution

(a) Set $z = a + bi$ with a and b real. Substitute to get

$$(a + bi)^2 + 3 + 4i = (a^2 - b^2 + 3) + (2ab + 4)i = 0$$

This yields the two equations:

$$a^2 - b^2 + 3 = 0 \qquad (1)$$

and

$$2ab + 4 = 0. \qquad (2)$$

From (2) we have $b = -2/a$ and substituting this in (1) yields

$$a^2 - \left(\frac{-2}{a}\right)^2 + 3 = 0$$

or

$$a^4 + 3a^2 - 4 = 0. \qquad (3)$$

Equation (3) is a quadratic in a^2 which has roots (by the quadratic formula)

$$a^2 = \frac{-3 \pm \sqrt{3^2 - 4(-4)}}{2} = \frac{-3 \pm 5}{2}.$$

Since a is real, a^2 must be non-negative and hence $a^2 = 1$. Thus $a = \pm 1$, $b = \mp 2$ and there are two solutions, $1 - 2i$ and $-1 + 2i$.

(b) Using the quadratic formula we find

$$z = \frac{-1 \pm \sqrt{1 - 4(1 + i)}}{2} = \frac{-1 \pm \sqrt{-3 - 4i}}{2}.$$

The expression $\pm\sqrt{-3 - 4i}$ represents the two numbers whose square is $-3 - 4i$, found in part (a). So substituting from part (a) we see that the required solutions are $-i$ and $-1 + i$.

4. For all complex numbers z_1 and z_2, prove that

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, (b) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$, (c) $\overline{z_1} = z_1$ if and only if z_1 is real,

and (d) $\overline{\left(\frac{1}{z_1}\right)} = \frac{1}{\overline{z_1}}$ for $z_1 \neq 0$.

Solution

Let $z_1 = a + ib$ and $z_2 = c + id$, where a, b, c, d are real numbers. Thus

$$z_1 + z_2 = a + c + i(b + d), \quad z_1 z_2 = ac - bd + i(ad + bc).$$

We then have

(a) $\overline{z_1 + z_2} = \overline{a + c + i(b + d)} = a + c - i(b + d) = a - ib + c - id = \overline{z_1} + \overline{z_2}$,

(b) $\overline{z_1 z_2} = \overline{ac - bd + i(ad + bc)} = ac - bd - i(ad + bc) = (a - ib)(c - id) = \overline{z_1}\ \overline{z_2}$,

and

(c) $\overline{z_1} = z_1$ if and only if $a + ib = a - ib$, that is, if and only if $b = -b$. Thus $b = 0$ and z_1 is real.

(d) As $z_1\left(\frac{1}{z_1}\right) = 1$, results (b) and (c) give

$$\overline{z_1\left(\frac{1}{z_1}\right)} = \overline{z_1}\ \overline{\left(\frac{1}{z_1}\right)} = \overline{1} = 1.$$

Therefore

$$\overline{\left(\frac{1}{z_1}\right)} = \frac{1}{\overline{z_1}}.$$

5. Sketch the following sets in the complex plane.
(*Hint*: Note that $|z-c|$ is the distance between z and c in the complex plane, hence $|z+c| = |z-(-c)|$ is the distance between z and $-c$.)

(a) $\{z \in \mathbb{C} \mid |z+i| = 5\}$,

(b) $\{z \in \mathbb{C} \mid \frac{1}{2} \leq |z+i| < 1\}$.

(c) $\{ z \in \mathbb{C} \mid |z| \leq 3 \}$

(d) $\{ z \in \mathbb{C} \mid |z+i| > 2 \}$

(e) $\{ z \in \mathbb{C} \mid \text{Re } z < -1 \}$

(f) $\{ z \in \mathbb{C} \mid \text{Im } z \geq -1 \}$

(g) $\{ z \in \mathbb{C} \mid |z-i| \leq |z-1| \}$

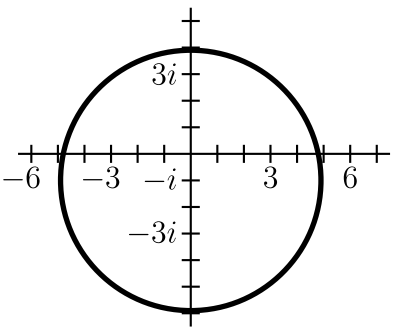
(h) $\{ z \in \mathbb{C} \mid \left| \frac{z-1}{z-2} \right| \leq 3 \}$

(i) $\{ z \in \mathbb{C} \mid \text{Im}(2z - \overline{z}(1+i)) = 0$ and $\text{Re}(2z - \overline{z}(1+i)) < 4 \}$

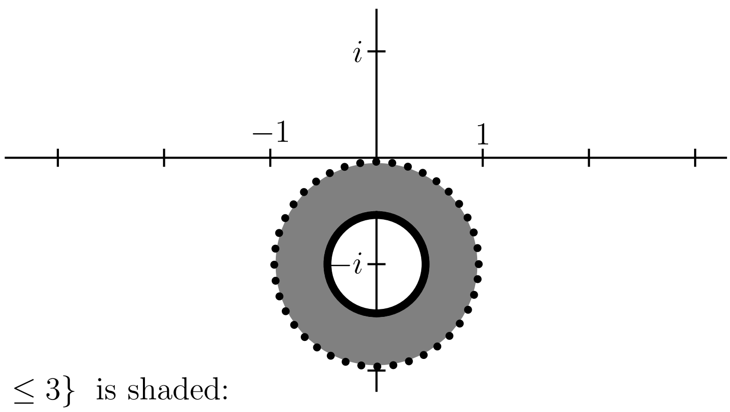
(j) $\{ z \in \mathbb{C} \mid \text{Im}(z^2) < \text{Re } z \}$

Solution

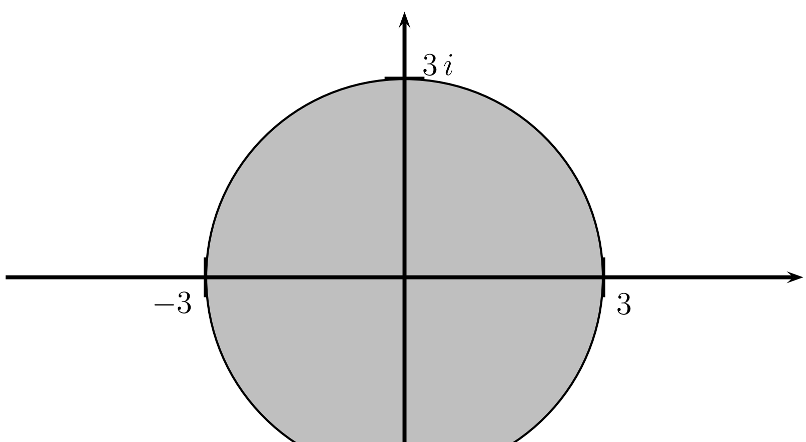
(a) The sketch of $\{z \in \mathbb{C} \mid |z+i| = 5\}$:



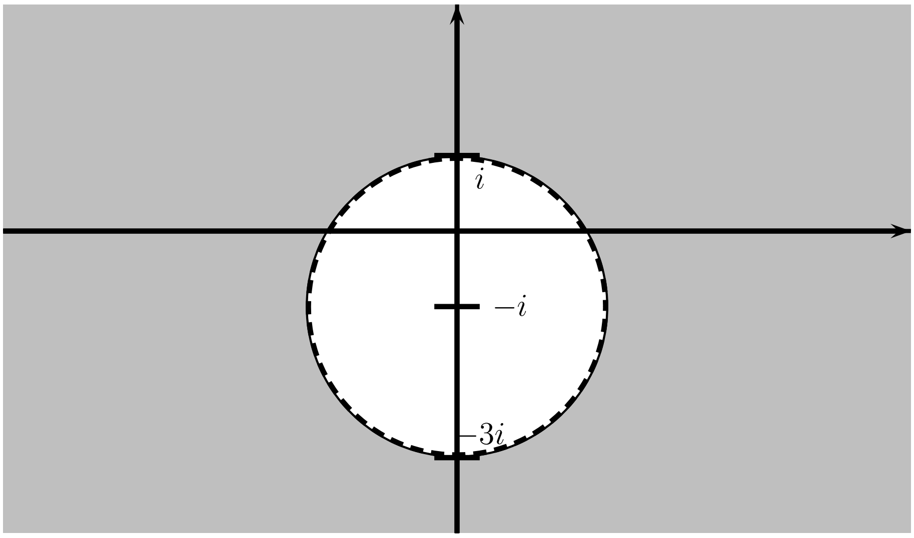
(b) The sketch of $\{z \in \mathbb{C} \mid \frac{1}{2} \leq |z+i| < 1\}$:



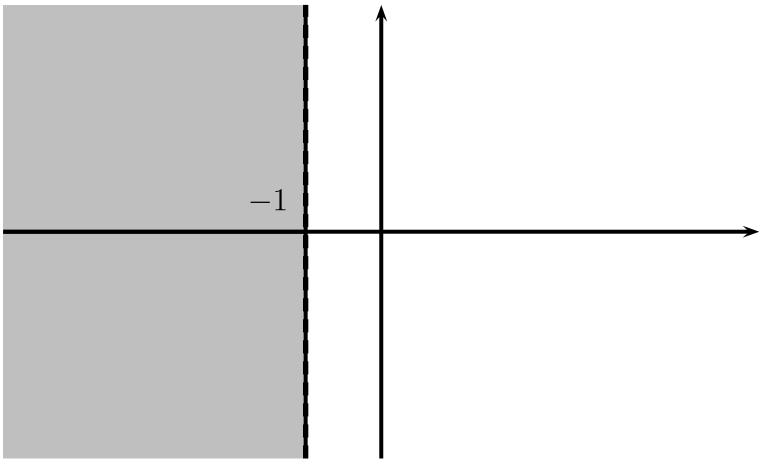
(c) $\{z \in \mathbb{C} \mid |z| \leq 3\}$ is shaded:



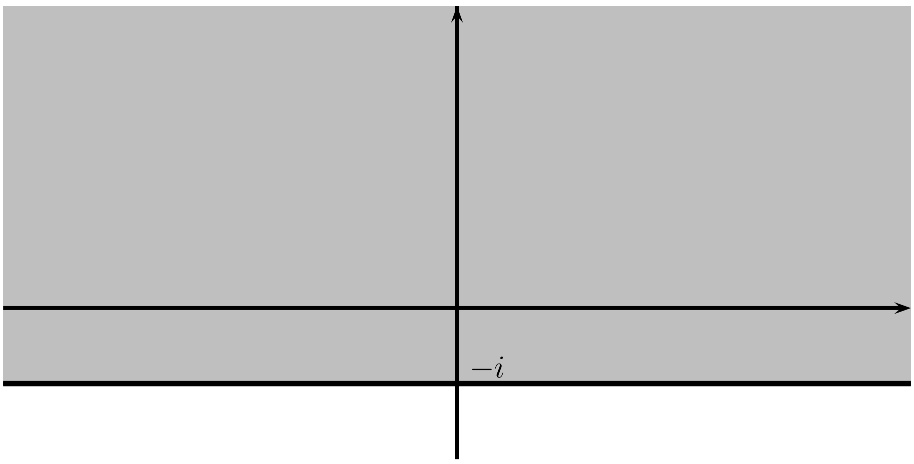
(d) $\{z \in \mathbb{C} \mid |z+i| > 2\}$ is shaded. It doesn't include the circle boundary and extends outwards from the circle in all directions, without bound.



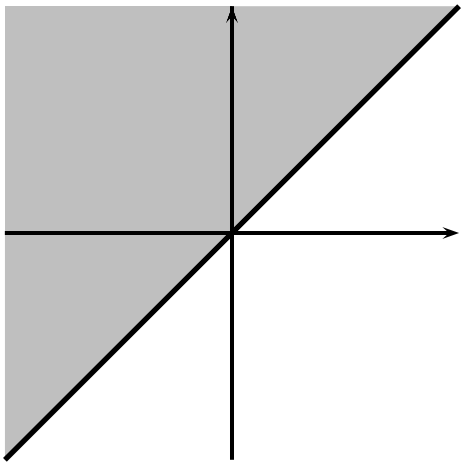
(e) $\{z \in \mathbb{C} \mid \text{Re}\, z < -1\}$ is shaded (not including the broken line):



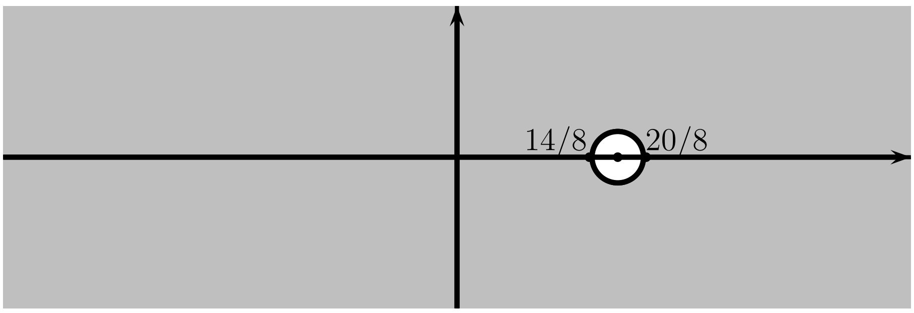
(f) $\{z \in \mathbb{C} \mid \text{Im}\, z \geq -1\}$ is shaded:



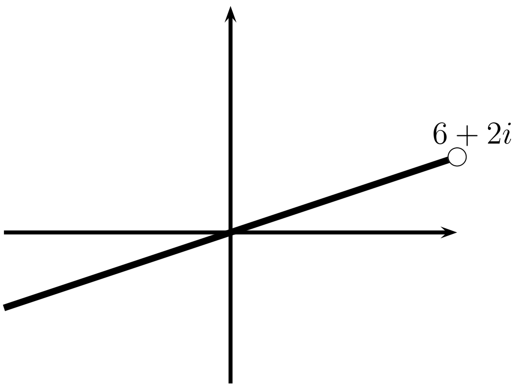
(g) $\{z \in \mathbb{C} \mid |z - i| \leq |z - 1|\}$ is shaded:



(h) $\{z \in \mathbb{C} \mid \left|\frac{z-1}{z-2}\right| \leq 3\}$ is shaded (it includes all the complex plane except for those complex numbers within $\frac{3}{8}$ of the real number $\frac{17}{8}$). To see this, rewrite the condition as $|z-1|^2 \leq 9|z-2|^2$, substitute $z = x + iy$ and simplify.



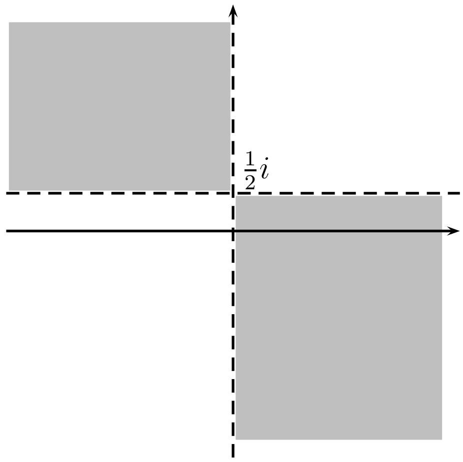
(i) Writing $z = x + iy$ where x and y are real, we see that $2z - \overline{z}(1+i) = x - y + i(3y - x)$. This has imaginary part zero if and only if $3y = x$. Thus the required complex numbers z are those numbers $z = 3y + iy = y(3+i)$ where $3y - y = 2y < 4$, that is, $y < 2$. This is the open half-line shown in the diagram.



(j) Let $z = x + iy$. Then $z^2 = (x^2 - y^2) + 2xyi$, so the condition in the question becomes $2xy < x$ or equivalently $x(2y - 1) < 0$. This is satisfied exactly when

$$x < 0,\ y > \frac{1}{2} \textbf{ or } x > 0,\ y < \frac{1}{2}.$$

So the set of solutions consists of the following two regions (dotted boundary lines not included):



6. (a) Write the following in *polar* form, $z = r(\cos\theta + i\sin\theta) = r \operatorname{cis} \theta$:

(*i*) $1 + i$ (*ii*) $1 + \sqrt{3}i$
(*iii*) $3\sqrt{3} + 3i$

(b) Using your answers to the previous part, find the following, expressing your answers first in polar then Cartesian (standard) form.

(*i*) $(1 + i)^{11}$ (*ii*) $(1 + \sqrt{3}i)^7$
(*iii*) $(3\sqrt{3} + 3i)^3$ (*iv*) $\dfrac{1 + i}{1 + \sqrt{3}i}$
(*v*) $\dfrac{3\sqrt{3} + 3i}{1 + i}$ (*vi*) $\dfrac{1 + \sqrt{3}i}{3\sqrt{3} + 3i}$

Solution

(a) (*i*) $\sqrt{2} \operatorname{cis} \dfrac{\pi}{4}$ (*ii*) $2 \operatorname{cis} \dfrac{\pi}{3}$
(*iii*) $6 \operatorname{cis} \dfrac{\pi}{6}$

(b) (*i*) $32\sqrt{2} \operatorname{cis} \dfrac{11\pi}{4} = 32(-1 + i)$

(*ii*) $128 \operatorname{cis} \dfrac{7\pi}{3} = 64(1 + \sqrt{3}\, i)$

(*iii*) $216 \operatorname{cis} \dfrac{\pi}{2} = 216\, i$

(*iv*) $\dfrac{1}{\sqrt{2}} \operatorname{cis} (-\dfrac{\pi}{12}) \approx \dfrac{1}{\sqrt{2}}(0.97 - 0.26\, i)$

(*v*) $\dfrac{6}{\sqrt{2}} \operatorname{cis} (-\dfrac{\pi}{12}) \approx \dfrac{6}{\sqrt{2}}(0.97 - 0.26\, i)$

(*vi*) $\frac{1}{3}\operatorname{cis}\frac{\pi}{6} = \frac{1}{2\sqrt{3}} + \frac{1}{6}i$

7. Prove the *triangle inequality* $|z_1 + z_2| \leq |z_1| + |z_2|$, for all $z_1, z_2 \in \mathbb{C}$.

Solution

We give an algebraic proof. Let $z_1 = a+ib$ and $z_2 = c+id$, where a, b, c, d are real numbers. The inequality we need to prove is

$$\sqrt{(a+c)^2 + (b+d)^2} \leq \sqrt{a^2+b^2} + \sqrt{c^2+d^2}.$$

Because both sides are nonnegative, it is equivalent to prove that the square of the left-hand side $\leq$ the square of the right-hand side:

$$(a+c)^2 + (b+d)^2 \leq a^2 + b^2 + c^2 + d^2 + 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}.$$

After tidying up, this becomes:

$$ac + bd \leq \sqrt{a^2+b^2}\sqrt{c^2+d^2}.$$

Since the right-hand side is nonnegative, it suffices to prove that the square of the left-hand side $\leq$ the square of the right-hand side:

$$a^2c^2 + b^2d^2 + 2abcd \leq (a^2+b^2)(c^2+d^2).$$

After tidying up, this becomes:

$$0 \leq a^2d^2 + b^2c^2 - 2abcd,$$

which is true because the right-hand side is $(ad-bc)^2$. Having ensured that each inequality in this proof implied the one before, we are justified in concluding that the original triangle inequality is also true. [Of course, the name 'triangle inequality' refers to the geometric interpretation: $|z_1 + z_2|$, $|z_1|$, and $|z_2|$ are the lengths of the three sides of the triangle (possibly degenerate) with vertices 0, z_1, and $z_1 + z_2$ in the complex plane. So if you assume the fact that the length of one side of a triangle is less than the sum of the other two side-lengths, the inequality follows. However, this fact about triangles arguably depends on some algebraic proof which is no easier than the one we have just done.]

8. If the complex number z is imagined as a point in the complex plane, then its conjugate $\overline{z}$ is the point obtained from z by reflecting in the real axis. What are the complex numbers obtained from z by the following geometric transformations?

(a) $180°$ rotation about 0.

(b) Reflection in the imaginary axis.

(c) $45°$ clockwise rotation about 0.

(d) Reflection in the line $y = x$.

Solution

(a) $180°$ rotation about 0 takes the point with co-ordinates (x, y) to the point with co-ordinates $(-x, -y)$. So it takes z to $-z$.

(b) Reflection in the imaginary axis takes $z = x+iy$ to $-\overline{z} = -x+iy$ (the real part changes sign, and the imaginary part stays the same).

(c) $45°$ clockwise rotation about 0 sends $z = r\operatorname{cis}\theta$ to $r\operatorname{cis}(\theta - \frac{\pi}{4})$. By the rule for multiplying complex numbers in polar form, this is the same as multiplying by $\operatorname{cis}(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$. So the answer is $z\operatorname{cis}(-\frac{\pi}{4})$.

(d) Reflection in the line $y = x$ takes the point with co-ordinates (x, y) to the point with co-ordinates (y, x). So it takes $z = x+iy$ to $i\overline{z} = y+ix$. Another way to see this is to carry out this reflection by a sequence of other rotations and reflections. For instance, the line we want to reflect in is such that if we rotate it 45° clockwise, it becomes the real axis. So we can carry out the reflection in three steps: first a 45° clockwise rotation taking z to $z \operatorname{cis}(-\frac{\pi}{4})$; then a reflection in the real axis, taking this to $\overline{z \operatorname{cis}(-\frac{\pi}{4})} = \overline{z} \operatorname{cis}(\frac{\pi}{4})$; then a 45° anticlockwise rotation, taking this to $\overline{z} \operatorname{cis}(\frac{\pi}{4} + \frac{\pi}{4}) = i\overline{z}$.

Extra Questions

9. Express $\cos 5\theta$ and $\sin 5\theta$ in terms of $\cos\theta$ and $\sin\theta$, respectively.
(*Hint*: Recall the binomial expansion:

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Use this as well as de Moivre's Theorem to expand $(\cos\theta + i\sin\theta)^5$.)

Solution

Expanding $(\cos\theta + i\sin\theta)^5$ yields

$$\begin{aligned}(\cos\theta + i\sin\theta)^5 &= (\cos\theta)^5 + 5(\cos\theta)^4 i\sin\theta + 10(\cos\theta)^3(i\sin\theta)^2 \\ &\qquad + 10(\cos\theta)^2(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5 \\ &= (\cos\theta)^5 + 5(\cos\theta)^4\sin\theta\, i - 10(\cos\theta)^3(\sin\theta)^2 \\ &\qquad - 10(\cos\theta)^2(\sin\theta)^3\, i + 5\cos\theta(\sin\theta)^4 + (\sin\theta)^5\, i.\end{aligned}$$

On the other hand, $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$.
Equating real and imaginary parts of each expression yields

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

and

$$\begin{aligned}\sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\ &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.\end{aligned}$$

10. (a) Let r be a real constant greater than 2. The set $\{z \in \mathbb{C} \mid |z + 1| + |z - 1| = r\}$ is a curve in the plane. Describe it and then find its equation in terms of x and y.

(b) Next, assume that $-2 < r < 2$ and $r \neq 0$.
Describe the curve $\{z \in \mathbb{C} \mid |z + 1| - |z - 1| = r\}$ and find its equation.

Solution

The condition which defines the set in part (a) can be interpreted geometrically as "the distance from z to -1 plus the distance from z to 1 is a constant which is greater than 2". Therefore the set in this part is an ellipse. The (real) equation of the ellipse is $\dfrac{x^2}{(\frac{r}{2})^2} + \dfrac{y^2}{(\frac{r}{2})^2 - 1} = 1$. The interpretation of the set in part (b) is done in a similar fashion.

This time we have a hyperbola with equation $\dfrac{x^2}{(\frac{r}{2})^2} - \dfrac{y^2}{1-(\frac{r}{2})^2} = 1.$

[These results can be obtained by setting $z = x + iy$ and transforming the given conditions into Cartesian equations in x and y. This process will be a test of your skill in algebraic manipulation.]

Solution to Question 1

1. (a) $(2+3i)+(4-5i) = 6-2i$; (b) $(1+i)(1-i) = 1^2 - i^2 = 1+1 = 2$;

(c) $(2+3i)-(4-5i) = -2+8i$; (d) $\dfrac{1+i}{1-i} = \dfrac{(1+i)(1+i)}{1^2-i^2} = \dfrac{(1+i)^2}{2} = i$;

(e) $\dfrac{1+2i}{3-4i} = \dfrac{1}{5}(-1+2i)$; (f) $(1+i)^2 = 2i$;

(g) $i^9 = i^8 i = i$; (h) $i^{123} - 4i^8 - 4i = -4-5i$.