

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Express the following complex numbers in Cartesian form:

- | | |
|--|---|
| (a) $2 \operatorname{cis} \frac{\pi}{4}$ | (b) $-4 \operatorname{cis} \frac{\pi}{3}$ |
| (c) $\operatorname{cis} \frac{\pi}{2} \operatorname{cis} \frac{\pi}{3} \operatorname{cis} \frac{\pi}{6}$ | (d) $e^{-i\pi}$ |
| (e) $e^{\ln 2 + i\pi}$ | (f) $e^{1+i} e^{1-i} e^{-2-i\pi}$ |

Questions for the tutorial

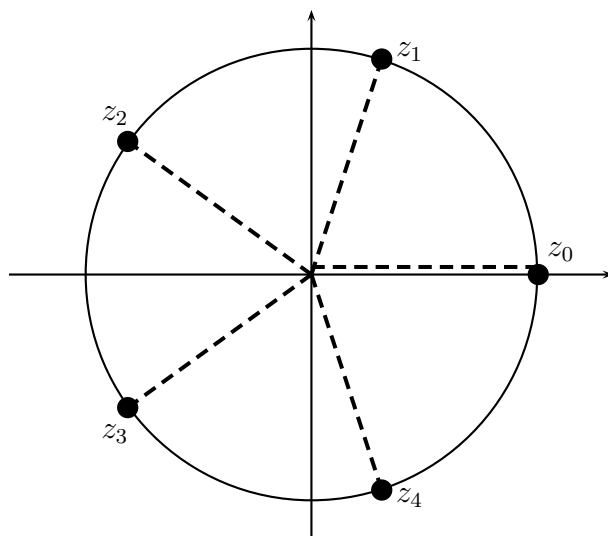
2. Solve the following equations (leaving your answers in polar form) and plot the solutions in the complex plane:
- $z^5 = 1$
 - $z^6 = -1$
 - $z^3 + i = 0$
 - $z^4 = 8\sqrt{2} + 8\sqrt{2}i$
 - $z^5 + z^3 - z^2 - 1 = 0$, given that $z = i$ is a solution.

Solution

- (a) Write $z = r \operatorname{cis} \theta$ and write 1 in polar form, $1 = 1 \operatorname{cis} 0$. Then $z^5 = 1 \iff r^5 \operatorname{cis} 5\theta = 1 \operatorname{cis} 0$. Equating moduli, we find that $r^5 = 1$, so $r = 1$. Comparing arguments, we find that $\theta = \frac{2k\pi}{5}$, for some $k \in \mathbb{Z}$. As k takes the five values 0, 1, 2, 3, 4, we get all five 5th roots of unity, namely

$$z_0 = 1, \quad z_1 = \operatorname{cis} (2\pi/5), \quad z_2 = \operatorname{cis} (4\pi/5), \quad z_3 = \operatorname{cis} (6\pi/5), \quad z_4 = \operatorname{cis} (8\pi/5).$$

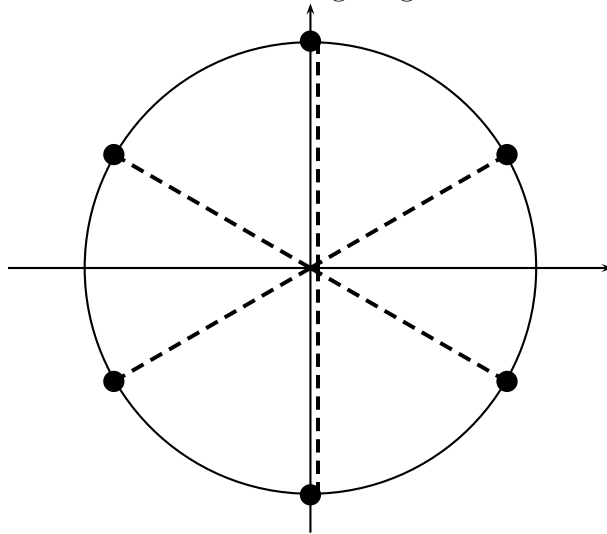
All solutions have modulus 1 and so lie on a unit circle, centred at the origin. The solutions (represented by the dots on the circle) are equally spaced at angles of $2\pi/5$.



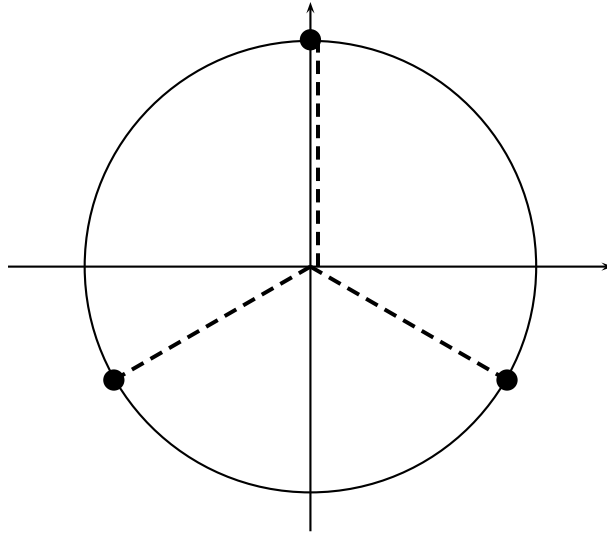
(b) Observe that $z^6 = -1 = \text{cis } \pi$. Therefore the six roots are:

$$z = \text{cis } (\pi/6), \text{ cis } (\pi/2), \text{ cis } (5\pi/6), \text{ cis } (-5\pi/6), \text{ cis } (-\pi/2), \text{ cis } (-\pi/6).$$

They are illustrated on the following diagram.

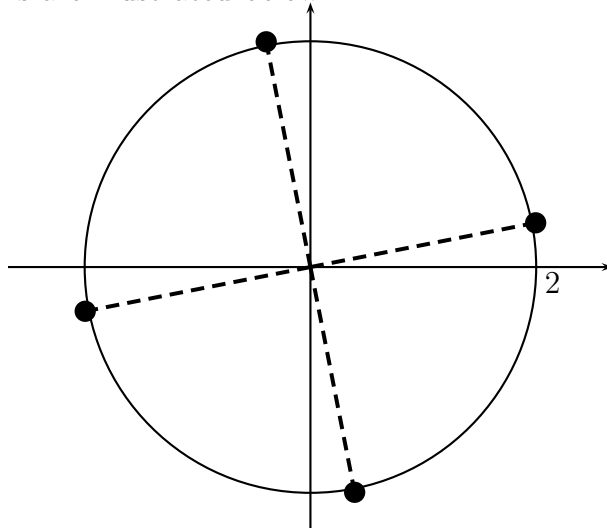


(c) $z^3 = -i = \text{cis } (-\pi/2) \iff z = \text{cis } (-\pi/6), \text{ cis } (\pi/2), \text{ cis } (-5\pi/6)$. The solutions are illustrated below.



(d) $z^4 = 8\sqrt{2} + 8\sqrt{2}i = 16 \text{cis } \pi/4 \iff z = 2 \text{cis } (\pi/16), 2 \text{cis } (9\pi/16), 2 \text{cis } (-15\pi/16), 2 \text{cis } (-7\pi/16)$.

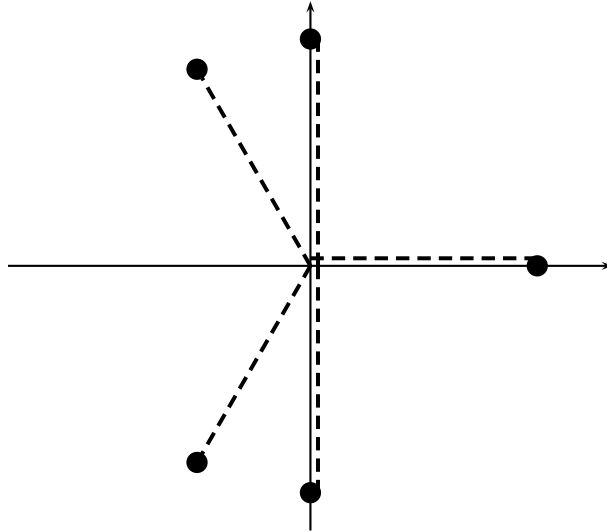
The solutions are illustrated below.



- (e) Given that $z = i$ is one solution, $z^5 + z^3 - z^2 - 1 = 0$ has the complex conjugate pair $z = \pm i$ as solutions, so $(z - i)(z + i) = z^2 + 1$ is a factor of the left hand side. Observe that

$$z^5 + z^3 - z^2 - 1 = z^3(z^2 + 1) - (z^2 + 1) = (z^2 + 1)(z^3 - 1) = 0.$$

Hence the solutions of the original polynomial equation are $\pm i = \text{cis}(\pm\pi/2)$ together with the three cube roots of 1 (which are $\text{cis} 0$ and $\text{cis}(\pm 2\pi/3)$). The solutions all have modulus 1 and are illustrated below.



3. The complex sine and cosine functions are defined by the formulas

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C}.$$

- (a) Show that when z is real ($z = x$), $\sin z$ and $\cos z$ reduce to the familiar real sine and cosine functions.
 (b) Show that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$.
 (c) Is it true that $|\sin z| \leq 1$ and $|\cos z| \leq 1$, for all $z \in \mathbb{C}$? (*Hint*: You know these are true when z is real. See what happens when z is purely imaginary, $z = iy$.)

Solution

- (a) If $z = x$, then

$$\begin{aligned} \sin z &= \frac{e^{ix} - e^{-ix}}{2i} \\ &= \frac{\cos x + i \sin x - (\cos x - i \sin x)}{2i} \\ &= \frac{2i \sin x}{2i} \\ &= \sin x. \end{aligned}$$

A similar result hold for $\cos z$.

- (b)

$$\sin^2 z + \cos^2 z = \frac{e^{2iz} - 2 + e^{-2iz}}{-4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} = 1.$$

- (c) These are not true statements. When $z = iy$, we have

$$|\sin(iy)| = \left| \frac{e^{-y} - e^y}{2i} \right| = \frac{1}{2} |e^y - e^{-y}|.$$

In particular, $|\sin(i)| = \frac{e - e^{-1}}{2} > 1$. Actually, $|\sin(iy)|$ can be made as large as we please by taking sufficiently large positive y (because e^y becomes arbitrarily large and dominates e^{-y}). A similar result holds for the complex cosine function.

4. Find all solutions of the following equations:

- (a) $e^z = i$ (b) $e^z = -10$
(c) $e^z = -1 - i\sqrt{3}$ (d) $e^{2z} = -i$

Solution

(a) Let $z = x + iy$. Recall that by definition of the complex exponential,

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \operatorname{cis}(y).$$

So the equation we want to solve is

$$e^x \operatorname{cis}(y) = i = 1 \operatorname{cis}(\pi/2).$$

Equating moduli, we find that $e^x = 1$, and so $x = 0$. Comparing arguments, we find that

$$y = \arg(e^z) = \arg(i) = \pi/2 + 2k\pi,$$

for some $k \in \mathbb{Z}$. Hence there are infinitely many solutions: $z = i(\frac{\pi}{2} + 2k\pi)$, $k \in \mathbb{Z}$.

- (b) There are infinitely many solutions: $z = \ln 10 + i(\pi + 2k\pi)$, $k \in \mathbb{Z}$.
(c) There are infinitely many solutions: $z = \ln 2 + i(-\frac{2\pi}{3} + 2k\pi)$, $k \in \mathbb{Z}$.
(d) There are infinitely many solutions: $z = i(-\frac{\pi}{4} + k\pi)$, $k \in \mathbb{Z}$.

5. Sketch and describe the following sets and their images under the function $z \mapsto z^2$.

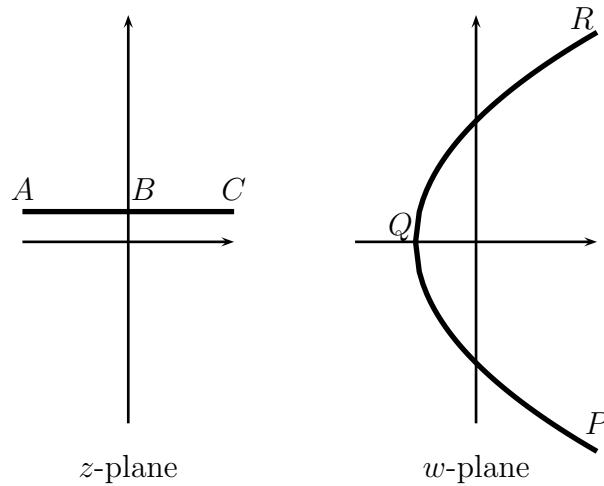
- (a) The set of all points of the form $z = x + 2i$.
(b) The set of all points of the form $z = x + 2xi$.
(c) The set of all points on the upper half of the unit circle centred at the origin, that is, points z with polar coordinates (r, θ) such that $r = 1$ and $0 \leq \theta \leq \pi$.
(d) The set of all points on the unit circle centred at the origin, that is, points z with polar coordinates (r, θ) such that $r = 1$ and $-\pi < \theta \leq \pi$.

Solution

(a) Let $w = z^2 = (x + 2i)^2 = x^2 - 4 + 4xi$. Remember that x can take any real value. If we write $w = u + iv$ then $u = x^2 - 4$ and $v = 4x$. We eliminate x from these two equations to obtain

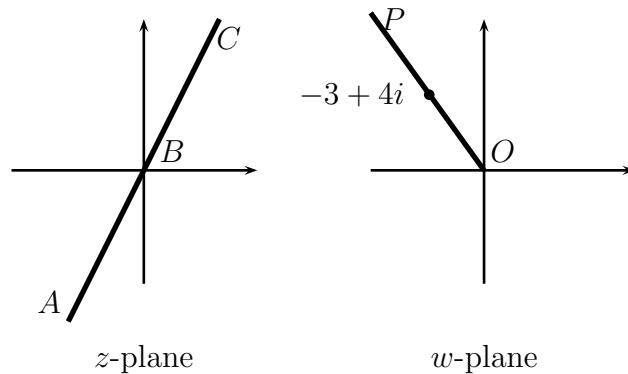
$$u = \left(\frac{v}{4}\right)^2 - 4 = \frac{v^2}{16} - 4.$$

Thus the image set in the w -plane is a parabola, symmetrical about the real axis, passing through the point -4 (denoted by Q in the diagram) and opening to the right.

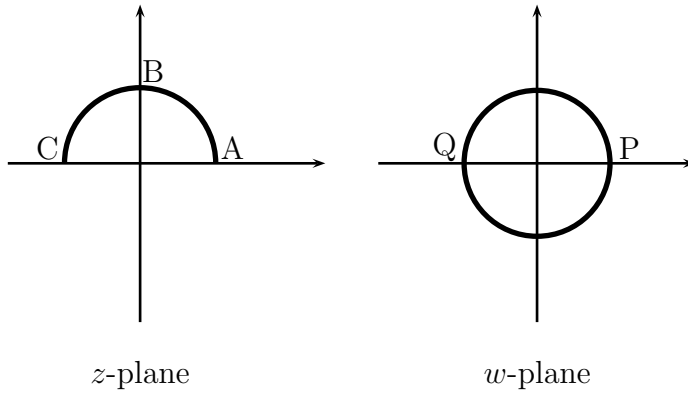


To see how the parabola is traced out as z varies, first think of x as taking very large negative values. Then v is also large and negative, while u is large and positive (we are at the lower right part of the parabola). Now let x increase. When $x = 0$, $v = 0$ and $u = -4$, corresponding to the apex of the parabola. When x is large and positive, so are both v and u (we are now at the upper right part of the parabola). That is, as z moves from A to B to C , w moves from P to Q to R .

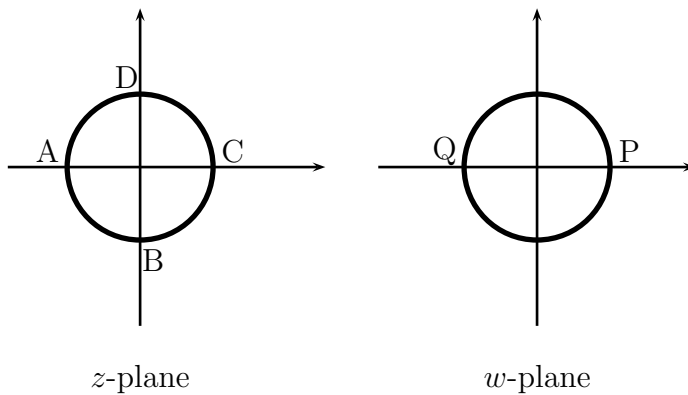
- (b) Let $w = z^2 = (x + 2xi)^2 = x^2(-3 + 4i)$. All values of w have the same principal argument. As x varies from very large negative numbers through zero to very large positive numbers, the modulus of w (namely $5x^2$) varies from very large, to zero, to very large again. (As z moves from A to B to C , w moves from P to O to P .) The image of the given set in the z -plane is a half-line traced twice in the w -plane, as shown in the following diagram.



- (c) As z has polar coordinates $(1, \theta)$, where $0 \leq \theta \leq \pi$, we see that $w = z^2$ has polar coordinates $(1, \phi)$, where $0 \leq \phi \leq 2\pi$. The image of the semi-circle in the z -plane is therefore the full unit circle in the w -plane. As z moves anticlockwise from A to B to C , w moves anticlockwise from P to Q (along the upper half circle) and then from Q to P (along the lower half circle).



(d) This time, the polar coordinates of z are $(1, \theta)$, where $-\pi < \theta \leq \pi$. Therefore the polar coordinates of w are $(1, \phi)$, where $-2\pi < \phi \leq 2\pi$. The image is again the unit circle. As z moves anticlockwise around the unit circle once (from just under A to B to C to D to A), w moves anticlockwise around the unit circle twice (from just above P to Q (upper half), Q to P (lower half), P to Q (upper half) and Q to P (lower half)).

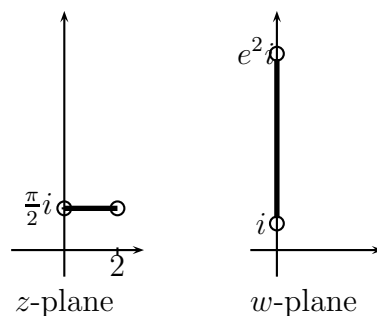


6. Sketch the following sets and their images under the function $z \mapsto e^z$:

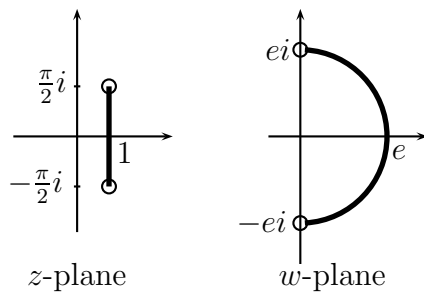
- (a) $\{z = x + iy \in \mathbb{C} \mid 0 < x < 2, y = \frac{\pi}{2}\}$;
- (b) $\{z \in \mathbb{C} \mid x = 1, |y| < \frac{\pi}{2}\}$;
- (c) $\{z \in \mathbb{C} \mid x < 0, \frac{\pi}{3} < y < \pi\}$;
- (d) $\{z = (1 + i)t \mid t \in \mathbb{R}\}$.

Solution

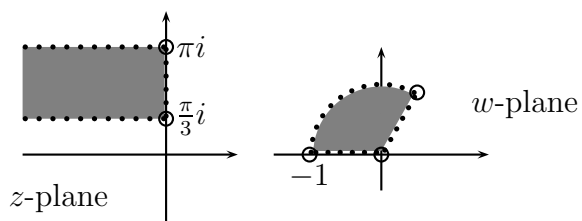
(a) Let $w = e^z$. As $z = x + i\frac{\pi}{2}$, we have $|w| = |e^z| = e^x$ (hence $1 < |w| < e^2$) and $\text{Arg}(w) = \frac{\pi}{2}$. The image of the given set is the set of points on the imaginary axis of the w -plane, as shown.



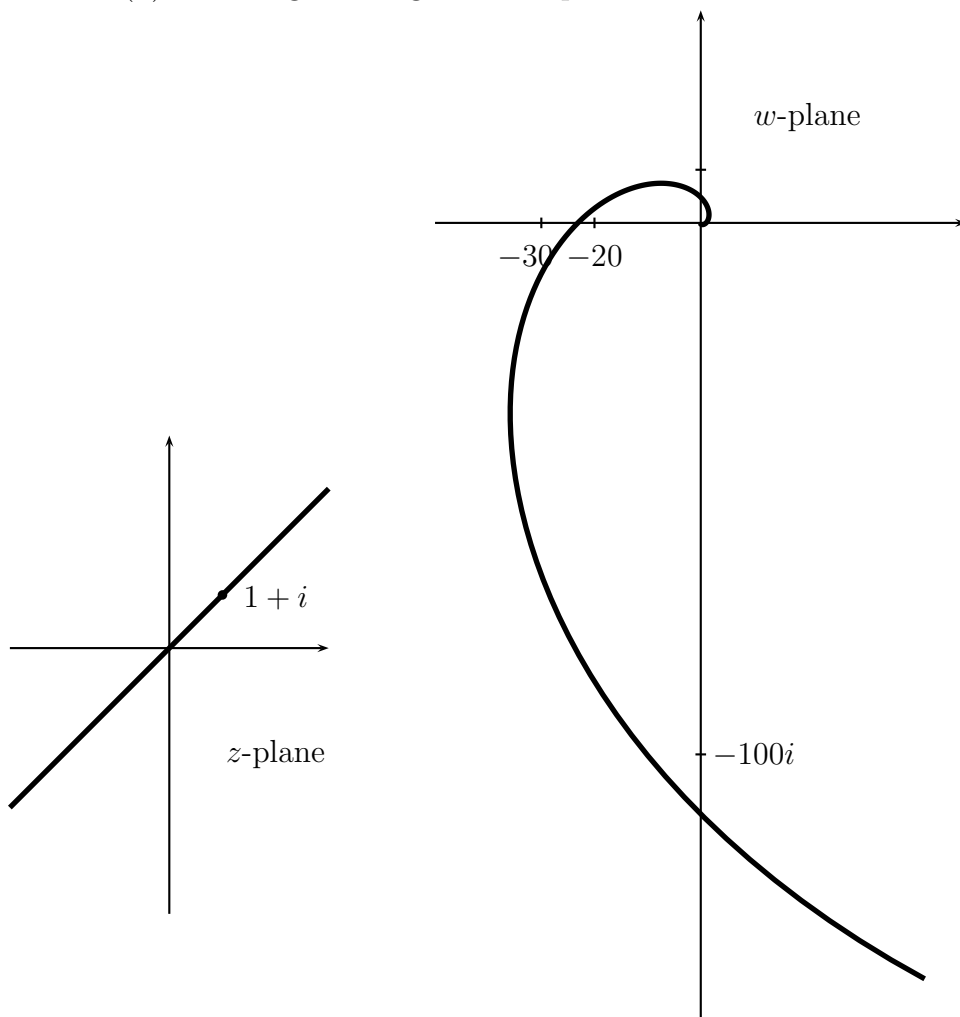
- (b) We know that $|w| = |e^z| = e^x = e$ (constant) and $\text{Arg}(w)$ lies between $-\pi/2$ and $\pi/2$. Hence the set of image points is a semi-circle.



- (c) Since $x < 0$ and $|w| = e^x$, we have $0 < |w| < 1$. Also, $\pi/3 < \text{Arg}(w) < \pi$. The image set is the interior of the sector of the unit circle, shown.



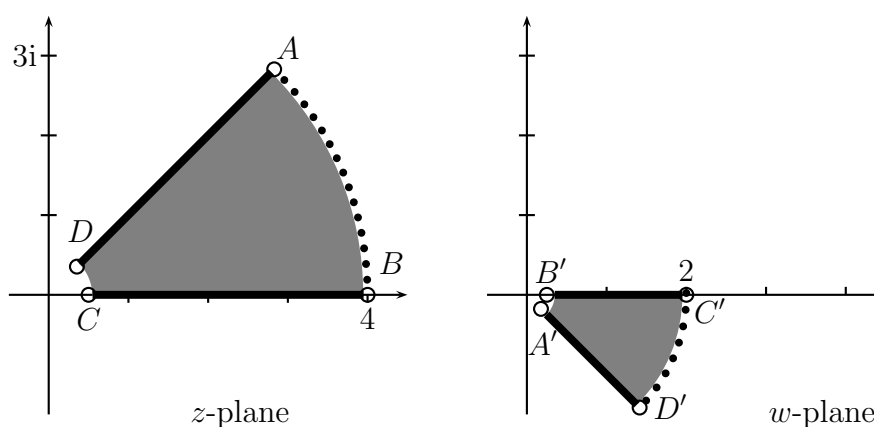
- (d) The image is a logarithmic spiral.



7. (a) Sketch the set $\{z \in \mathbb{C} \mid \frac{1}{2} < |z| < 4, 0 \leq \text{Arg}(z) \leq \frac{\pi}{4}\}$.
- (b) Sketch the image of the set in the w -plane under the function $z \mapsto w = \frac{1}{z}$.
- (c) An insect is crawling clockwise around the boundary of the set in the z -plane. Is its image crawling clockwise, or anticlockwise, in the w -plane? (If clockwise, we say the transformation is *orientation-preserving*; if anticlockwise, we say it is *orientation-reversing*.)
- (d) Now consider the function $z \mapsto w = \bar{z}$, the complex conjugate of z . Is it orientation-preserving or orientation-reversing?

Solution

- (a) The sketch of the original set in the z -plane is given below, on the left.



- (b) The sketch of the image of the set in the w -plane is given above, on the right.
- (c) The mapping is orientation-preserving. As the insect crawls from A to B to C to D, its image crawls from A' to B' to C' to D'.
- (d) The complex conjugate transformation can be thought of as reflection in the real axis and is orientation-reversing.

8. Find all solutions of the equation $e^{2z} - (1 + 3i)e^z + i - 2 = 0$.

Solution

Using the quadratic formula we find that this equation is equivalent to

$$e^z = \frac{(1 + 3i) \pm \sqrt{(1 + 3i)^2 - 4(i - 2)}}{2},$$

which simplifies to

$$e^z = \frac{1 + 3i \pm \sqrt{2i}}{2}.$$

As $(1 + i)^2 = 2i$, we see that the two numbers whose square is $2i$ are $\pm(1 + i)$, so

$$e^z = \frac{1 + 3i \pm (1 + i)}{2} = 1 + 2i \text{ or } i.$$

In polar form, $1 + 2i = \sqrt{5}e^{i\theta}$, where $\theta = \tan^{-1}(2) \approx 1.11$ radians, and $i = e^{i\pi/2}$. If $z = x + iy$ then

$$e^z = e^x e^{iy} = \sqrt{5}e^{i\theta} \text{ or } e^{i\pi/2}.$$

Hence $x = \frac{1}{2} \ln 5$ and $y = \theta + 2k\pi$, or $x = 0$ and $y = \pi/2 + 2k\pi$, for $k \in \mathbb{Z}$. There are infinitely many solutions, of the form

$$z = \frac{1}{2} \ln 5 + i(\tan^{-1}(2) + 2k\pi) \quad \text{or} \quad z = i(\pi/2 + 2k\pi).$$

Extra Questions

9. This question demonstrates that complex numbers can be useful in solving cubic equations, even when all the solutions are real.

- (a) Show that for any complex number w , there exists a nonzero complex number z such that $z + \frac{1}{z} = w$.
- (b) Use this substitution to solve the equation $w^3 - 3w - 1 = 0$.

Solution

(a) The equation $z + \frac{1}{z} = w$ becomes $z^2 - wz + 1 = 0$ after multiplying by z and rearranging.

This equation is a quadratic in z and has solutions $z = \frac{w \pm \sqrt{w^2 - 4}}{2}$. These solutions are nonzero, because $0^2 - w \cdot 0 + 1 = 1 \neq 0$. Hence we can divide by z to conclude that we have found solutions to the initial equation $z + \frac{1}{z} = w$.

(b) We know from (a) that any w can be written as $z + \frac{1}{z}$ for some nonzero $z \in \mathbb{C}$. When this substitution is made,

$$\begin{aligned} w^3 - 3w - 1 &= \left(z + \frac{1}{z}\right)^3 - 3\left(z + \frac{1}{z}\right) - 1 \\ &= z^3 + 3z^2\frac{1}{z} + 3z\frac{1}{z^2} + \frac{1}{z^3} - 3\left(z + \frac{1}{z}\right) - 1 \\ &= z^3 + \frac{1}{z^3} - 1, \end{aligned}$$

so the equation $w^3 - 3w - 1 = 0$ is equivalent to $z^3 + \frac{1}{z^3} - 1 = 0$, which is equivalent to $z^6 - z^3 + 1 = 0$. This is a quadratic in z^3 with solutions

$$z^3 = \frac{1 + \sqrt{3}i}{2} = \text{cis}\left(\frac{\pi}{3}\right) \quad \text{or} \quad z^3 = \frac{1 - \sqrt{3}i}{2} = \text{cis}\left(-\frac{\pi}{3}\right).$$

Since the solutions of $z^3 = \text{cis}\left(-\frac{\pi}{3}\right)$ are the inverses of the solutions of $z^3 = \text{cis}\left(\frac{\pi}{3}\right)$, and $z + \frac{1}{z}$ doesn't change when you replace z with its inverse, we need only consider the possibility that $z^3 = \text{cis}\left(\frac{\pi}{3}\right)$. As in Q2, we find that the three cube roots of $\text{cis}\left(\frac{\pi}{3}\right)$ are $\text{cis}\left(\frac{\pi}{9}\right)$, $\text{cis}\left(\frac{7\pi}{9}\right)$, and $\text{cis}\left(-\frac{5\pi}{9}\right)$. Now if $z = \text{cis}\theta$, then

$$z + \frac{1}{z} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) = 2\cos\theta,$$

so we have proved that the solutions to the original equation $w^3 - 3w - 1 = 0$ are $2\cos\frac{\pi}{9}$, $2\cos\frac{5\pi}{9}$, and $2\cos\frac{7\pi}{9}$. (Notice that these are all real numbers, but we needed the complex numbers to find them.)

10. Let n be a given positive integer. By a *primitive n th root of unity* we mean a solution η of $z^n = 1$ which has the property that its powers $\eta, \dots, \eta^{n-1}, \eta^n (= 1)$ are exactly the solutions of this equation in \mathbb{C} . For example, $e^{i\frac{2\pi}{n}}$ is a primitive n th root of unity.

(a) Find all primitive 6th roots of unity.

(b) Find all primitive 5th roots of unity.

(c) For which values of k , $0 \leq k \leq n - 1$, is $e^{i\frac{2\pi k}{n}}$ a primitive n th root of unity?

Solution

(a) Clearly, $\eta = e^{i\frac{2\pi}{6}}$ is a primitive 6-th root of unity, as its powers $\eta, \eta^2, \eta^3, \eta^4, \eta^5, \eta^6 (= 1)$ give the complete list of 6-th roots of unity, namely

$$\eta = e^{i\frac{2\pi}{6}}, \quad \eta^2 = e^{i\frac{4\pi}{6}}, \quad \eta^3 = e^{i\frac{6\pi}{6}}, \quad \eta^4 = e^{i\frac{8\pi}{6}}, \quad \eta^5 = e^{i\frac{10\pi}{6}}, \quad \eta^6 = e^{i\frac{12\pi}{6}} = 1.$$

However $\eta^2 = e^{i\frac{4\pi}{6}}$ is not a primitive 6-th root of unity: its distinct powers are $\eta^2, (\eta^2)^2 = \eta^4$ and $(\eta^2)^3 = \eta^6 = 1$. Similarly, η^3 is not a primitive root (its distinct powers are $\eta^3, \eta^6 = 1$) and η^4 is not a primitive root (its distinct powers are η^4, η^2 and 1). However, η^5 is a primitive root. Its powers are

$$\eta^5, \quad (\eta^5)^2 = \eta^4, \quad (\eta^5)^3 = \eta^3, \quad (\eta^5)^4 = \eta^2, \quad (\eta^5)^5 = \eta, \quad (\eta^5)^6 = 1.$$

Thus the only primitive 6th roots of unity are η and η^5 .

(b) With $\eta = e^{i\frac{2\pi}{5}}$, we find that the primitive 5th roots of unity are

$$\eta, \quad \eta^2, \quad \eta^3, \quad \eta^4.$$

(c) The n th root of unity $e^{i\frac{2\pi k}{n}}$ is a primitive n th root if, and only if, k is *relatively prime* to n . (Can you prove this?)

Solution to Question 1

1. (a) $2 \operatorname{cis} \pi/4 = \sqrt{2} + \sqrt{2}i$ (b) $-4 \operatorname{cis} \pi/3 = -2 - 2\sqrt{3}i$

(c) $\operatorname{cis} (\pi/2) \operatorname{cis} (\pi/3) \operatorname{cis} (\pi/6) = \operatorname{cis} \pi = -1$ (d) $e^{-i\pi} = \operatorname{cis} (-\pi) = -1$

(e) $e^{\ln 2 + i\pi} = e^{\ln 2} \operatorname{cis} \pi = -2$ (f) $e^{1+i} e^{1-i} e^{-2-i\pi} = e^{1+i+1-i-2-i\pi} = e^{-i\pi} = -1$