THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Tutorial Solutions Week 4** 2009

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

For each of the following functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, sketch its graph and decide whether the function is injective, whether it is surjective, and whether it is bijective.

(a) $f(x) = |x|$, (b) $f(x) = x^3 + 1$, (c) $f(x) = \begin{cases} -x^2, \ x < 0, \\ 0, \ 0 \le x \le 1, \\ x - 1, \ x > 1. \end{cases}$

Questions for the tutorial

2. Each formula below belongs to a function defined on some subset of $\mathbb{R}$. For each one, find the natural domain of the function (that is, the largest subset of $\mathbb{R}$ for which the rule makes sense) and the corresponding range. For the domain and range you found, decide if the function is a bijection and if so, find a formula for the inverse function.

(a) $f(x) = \dfrac{x-2}{x+2}$, (b) $f(x) = \sqrt{2+5x}$, (c) $f(x) = x|x| + 1$.

Solution

(a) The natural domain (also known as the domain of definition) of f is $\mathbb{R} \setminus \{-2\}$, that is, the set of all real numbers except -2. Observe that for any x, $\frac{x-2}{x+2} \neq 1$, and so 1 is not in the range. The number y is in the range if the equation $y = \dfrac{x-2}{x+2}$ has at least one solution for x in the domain. Rearranging this equation gives $x = \dfrac{2(1+y)}{1-y}$, showing that for any $y \neq 1$, there is one and only one x, namely $x = \dfrac{2(1+y)}{1-y}$, such that $y = \dfrac{x-2}{x+2}$. Hence the range of f is $\mathbb{R} \setminus \{1\}$ and the function

$$f : \mathbb{R} \setminus \{-2\} \longrightarrow \mathbb{R} \setminus \{1\}, \qquad f(x) = \frac{x-2}{x+2}$$

is a bijection. The inverse function is

$$g : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{-2\}, \qquad g(y) = \frac{2(1+y)}{1-y}.$$

(The graph of f is a hyperbola with vertical asymptote $x = -2$ and horizontal asymptote $y = 1$.)

(b) The natural domain is $[-\frac{2}{5}, \infty)$. The range of f is $[0, \infty)$. For any $y \ge 0$, there is one and only one x such that $y = \sqrt{2+5x}$, namely $x = \frac{1}{5}(y^2 - 2)$. Hence

$$f : [-\tfrac{2}{5}, \infty) \longrightarrow [0, \infty)$$

is a bijection and the inverse function is

$$g : [0, \infty) \to [-\tfrac{2}{5}, \infty), \qquad g(y) = \tfrac{1}{5}(y^2 - 2).$$

(c) The natural domain is $\mathbb{R}$. When $x \geq 0$, $f(x) = x^2 + 1$ and $f(x)$ takes all values in $[1, \infty)$. When $x \leq 0$, $f(x) = 1 - x^2$ and $f(x)$ takes all values in $(-\infty, 1]$. Hence the range of f is $\mathbb{R}$. The function is injective as it is an increasing function on each of the intervals $[0, \infty)$ and $(-\infty, 0]$, and therefore increasing on the whole of $\mathbb{R}$. Hence $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a bijection and its inverse function is $g : \mathbb{R} \longrightarrow \mathbb{R}$, where

$$g(y) = \begin{cases} -\sqrt{1-y}, & y < 1, \\ \sqrt{y-1}, & y \geq 1. \end{cases}$$

3. Explain why the functions given by the formulas and domains below are injective. Find their ranges and formulas for their inverses.

(a) $f(x) = x^2 + x, \ \ x \geq -\frac{1}{2}$.

(b) $g(x) = \sqrt[4]{x}, \ \ x \geq 0$.

(c) $h(x) = \dfrac{1+e^x}{1-e^x}, \ \ x \neq 0$.

(d) $f(x) = \ln(3 + \sqrt{x-4}), \ \ x \geq 5$.

Solution

(a) Because $x \mapsto x^2$ is an increasing function on $[0, \infty)$, $f(x) = (x + \frac{1}{2})^2 - \frac{1}{4}$ is an increasing function on the domain $[-1/2, \infty)$. It is therefore injective. As x runs over $[-1/2, \infty)$, $(x + \frac{1}{2})^2$ runs over $[0, \infty)$, so $f(x)$ runs over $[-1/4, \infty)$. Thus the range is $[-1/4, \infty)$. Solving the equation $y = x^2 + x$ for x gives $x = -\frac{1}{2} \pm \sqrt{y + \frac{1}{4}}$. As we are only interested in the case that $x \geq -\frac{1}{2}$, we take the positive square root. Thus we get the following rule for the inverse function:

$$f^{-1}(y) = -\tfrac{1}{2} + \sqrt{y + \tfrac{1}{4}}.$$

(b) $g(x)$ is injective by definition, because it is defined to be the inverse of the bijection $f : [0, \infty) \to [0, \infty)$ given by $f(y) = y^4$. (To spell out the proof of injectivity for this particular case, if we have $\sqrt[4]{x_1} = \sqrt[4]{x_2}$ then we must have $x_1 = x_2$, just by raising both sides to the fourth power.) So the range of $g(x)$ is $[0, \infty)$ and its inverse is

$$g^{-1}(y) = y^4.$$

(c) Note that $h(x) = -1 + \dfrac{2}{1-e^x}$. Because $x \mapsto e^x$ is an increasing function, $x \mapsto 1 - e^x$ is a decreasing function, so h is increasing on $(-\infty, 0)$ and also on $(0, \infty)$. It is not, however, increasing on the whole domain $\mathbb{R} \setminus \{0\}$: for example, $h(-1) > h(1)$. Nevertheless, h is injective, because it takes positive values on $(-\infty, 0)$ and negative values on $(0, \infty)$, so no horizontal line cuts the graph more than once. In fact, as x runs over $(-\infty, 0)$, $h(x)$ takes all values in $(1, \infty)$; and as x runs over $(0, \infty)$, $h(x)$ takes all values in $(-\infty, -1)$. So the range of h is $(-\infty, -1) \cup (1, \infty)$.
To find the inverse function, set $y = h(x) = -1 + \frac{2}{1-e^x}$. Rearranging gives $e^x = 1 - \dfrac{2}{y+1}$, so $x = \ln(1 - \dfrac{2}{y+1}) = \ln(\dfrac{y-1}{y+1})$. Thus, for $y < -1$ or $y > 1$, we get:

$$h^{-1}(y) = \ln\left(\frac{y-1}{y+1}\right).$$

(d) f is injective because it is strictly increasing on the given domain; this is because $\sqrt{x}$ is an increasing function and so is $\ln(x)$. As x runs over $[5, \infty)$, $\sqrt{x-4}$ runs over $[1, \infty)$, so $\ln(3 + \sqrt{x-4})$ runs over $[\ln(4), \infty)$. Thus the range is $[\ln(4), \infty)$. Solving the equation $y = \ln(3 + \sqrt{x-4})$ for x gives $x = (e^y - 3)^2 + 4$, so the formula for the inverse function is

$$f^{-1}(y) = (e^y - 3)^2 + 4.$$

4. For what values of the constants a, b, c (with $b \neq 0$) is the function with formula

$$f(x) = \frac{x-a}{bx-c}, \qquad \text{and domain } x \neq \frac{c}{b},$$

equal to its own inverse? (*Hint*: It may help to draw the graph.)

Solution

Using the fact that $b \neq 0$, we can rewrite the formula for $f(x)$ as follows:

$$f(x) = \frac{1}{b} + \frac{\frac{c-ab}{b^2}}{x - \frac{c}{b}}.$$

Notice that if $c = ab$, then this becomes the constant function $f(x) = \frac{1}{b}$, which is clearly not injective and hence has no inverse. So we must have $c \neq ab$. Then the graph of $f(x)$ is a hyperbola, with vertical asymptote at $x = \frac{c}{b}$, and horizontal asymptote at $y = \frac{1}{b}$. So the domain of f is $\mathbb{R} \setminus \{\frac{c}{b}\}$, and the range of f is $\mathbb{R} \setminus \{\frac{1}{b}\}$. Setting $y = f(x)$ and solving for x, we get a formula for the inverse function:

$$f^{-1}(y) = \frac{c}{b} + \frac{\frac{c-ab}{b^2}}{y - \frac{1}{b}},$$

which is another hyperbola, this time with domain $\mathbb{R} \setminus \{\frac{1}{b}\}$ and range $\mathbb{R} \setminus \{\frac{c}{b}\}$. In order to have $f = f^{-1}$, the domain of f must equal the domain of f^{-1} (which is the range of f), so we must have $c = 1$. Conversely, if $c = 1$ then the two formulas become the same, so f does equal its own inverse. To sum up, the answer to the question is that f is equal to its own inverse whenever $b \neq 0$, $c = 1$, and $a \neq 1/b$.

5. The function $\cosh : (0, \infty) \longrightarrow (1, \infty)$ is a bijection and the function $\cosh : (-\infty, 0) \longrightarrow (1, \infty)$ is also a bijection.

(a) Let $\cosh^{-1}$ denote the inverse function of the first-mentioned bijection. Show that $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$.

(b) How would the answer change if you used $(-\infty, 0)$ as the domain of cosh?

Solution

(a) Put $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$, so $2y = e^x + e^{-x}$. Thus $e^{2x} - 2ye^x + 1 = 0$, yielding

$$e^x = \frac{2y \pm \sqrt{4y^2 - 4}}{2} = y \pm \sqrt{y^2 - 1}\,,$$

so that $x = \ln(y \pm \sqrt{y^2 - 1})$. We must choose the positive sign to ensure that $x > 0$, because $y - \sqrt{y^2 - 1} < 1$ for $y > 1$. (To prove this, note that it is equivalent to $y - 1 < \sqrt{y^2 - 1}$, which is equivalent to $(y-1)^2 < y^2 - 1$ under the assumption $y > 1$, and this last inequality is easy.) Thus we get $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$ when the domain of cosh is $(-\infty, 0)$.

(b) If $x \in (-\infty, 0)$, we would take the negative sign, to obtain $\cosh^{-1} x = \ln(x - \sqrt{x^2 - 1})$.

6. Show that

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y, \text{ and}$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$

for all x, $y \in \mathbb{R}$.

Solution

(a)

$$
\begin{aligned}
&\cosh x \cosh y + \sinh x \sinh y \\
&\qquad = \left[\tfrac{1}{2}(e^x + e^{-x})\right]\left[\tfrac{1}{2}(e^y + e^{-y})\right] + \left[\tfrac{1}{2}(e^x - e^{-x})\right]\left[\tfrac{1}{2}(e^y - e^{-y})\right] \\
&\qquad = \tfrac{1}{4}\left[(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}) + (e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y})\right] \\
&\qquad = \tfrac{1}{4}(2e^{x+y} + 2e^{-x-y}) \\
&\qquad = \tfrac{1}{2}(e^{x+y} + e^{-x-y}) \\
&\qquad = \cosh(x+y)
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\sinh x \cosh y + \cosh x \sinh y \\
&\qquad = \left[\tfrac{1}{2}(e^x - e^{-x})\right]\left[\tfrac{1}{2}(e^y + e^{-y})\right] + \left[\tfrac{1}{2}(e^x + e^{-x})\right]\left[\tfrac{1}{2}(e^y - e^{-y})\right] \\
&\qquad = \tfrac{1}{4}\left[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})\right] \\
&\qquad = \tfrac{1}{4}(2e^{x+y} - 2e^{-x-y}) \\
&\qquad = \tfrac{1}{2}(e^{x+y} - e^{-(x+y)}) \\
&\qquad = \sinh(x+y)
\end{aligned}
$$

7. (a) Suppose that $g(x)$ is (strictly) increasing on the domain $D \subseteq \mathbb{R}$, i.e. if $x_1, x_2 \in D$ and $x_1 < x_2$, then $g(x_1) < g(x_2)$. Suppose that $f(x)$ is increasing on $E \subseteq \mathbb{R}$, and that $f(E) \subseteq D$. Prove that the composite function $g(f(x))$ is increasing on E.

(b) Using the result of the previous part, and the fact that e^x is increasing on the whole real line, prove that $\cosh x$ is increasing on $[0, \infty)$.

Solution

(a) We need to show that if $x_1, x_2 \in E$ and $x_1 < x_2$, then $g(f(x_1)) < g(f(x_2))$. But $f(x)$ is increasing on E, so $f(x_1) < f(x_2)$. We also know that $f(x_1), f(x_2) \in D$, because $f(E) \subseteq D$. Since $g(x)$ is increasing on D, we can conclude that $g(f(x_1)) < g(f(x_2))$, as required.

(b) By definition, $\cosh x = g(e^x)$ where $g(x) = \dfrac{x + x^{-1}}{2}$. To apply the result of the previous part, we take $E = [0, \infty)$. Certainly e^x is increasing on E, and when restricted to this domain its range is $D = [1, \infty)$. So all we need to check is that $g(x)$ is increasing on D. In other words, we need to show that if $1 \le x_1 < x_2$, then

$$\frac{x_1 + x_1^{-1}}{2} < \frac{x_2 + x_2^{-1}}{2}.$$

After multiplying by 2 and rearranging, this inequality becomes

$$(x_2 - x_1)(1 - x_1^{-1}x_2^{-1}) > 0,$$

which is true because both factors on the left-hand side are strictly positive. So we have proved that $\cosh x$ is increasing on $[0, \infty)$.

8. If $f : A \to B$ and $g : B \to C$ are injective functions, then the composite function $g \circ f : A \to C$ is also injective. In this question you are asked to construct a proof of this result, starting with the sentence:

"Let $a_1, a_2 \in A$, and assume that $a_1 \neq a_2$."

Complete the proof using five of the following eight phrases (in some order), together with appropriate logical connecting words (e.g. "because", "so", "therefore").

(1) f is injective
(2) g is injective
(3) $g \circ f$ is injective
(4) $b_1, b_2 \in B$
(5) $b_1 \neq b_2$
(6) $f(a_1) \neq f(a_2)$
(7) $g(b_1) \neq g(b_2)$
(8) $g(f(a_1)) \neq g(f(a_2))$

Solution

Phrases (4), (5), and (7) are not needed. The others can be ordered (1)–(6), (2)–(8), (3): for instance, the proof could run as follows. "Let $a_1, a_2 \in A$, and assume that $a_1 \neq a_2$. *Therefore* $f(a_1) \neq f(a_2)$, *because* f is injective. *Therefore* $g(f(a_1)) \neq g(f(a_2))$, *because* g is injective. *Therefore* $g \circ f$ is injective."
An alternative proof would use the contrapositive form of the definition of injectivity ("if the outputs are the same, the inputs must have been the same"). This proof would run as follows. "Let $a_1, a_2 \in A$, and assume that $(g \circ f)(a_1) = (g \circ f)(a_2)$. This means that $g(f(a_1)) = g(f(a_2))$. Therefore $f(a_1) = f(a_2)$, because g is injective. Therefore $a_1 = a_2$, because f is injective. Therefore $g \circ f$ is injective."

Extra Questions

9. Is the following statement true or false: "a function $f : \mathbb{R} \to \mathbb{R}$ is injective if and only if f is either strictly increasing or strictly decreasing"?
If you think it is true, give a proof. If you think it is false, give a counter-example.

Solution

It is true to say that if f is strictly increasing or strictly decreasing, then f is injective. For if x_1 and x_2 are distinct elements in the domain such that $x_1 < x_2$, then either $f(x_1) > f(x_2)$ (if f is decreasing) or $f(x_1) < f(x_2)$ (if f is increasing). In both cases, $f(x_1) \neq f(x_2)$. This shows that distinct inputs produce distinct outputs. However, the converse is not true. As a counter-example, consider the function f with domain $\mathbb{R}$ given by the formula

$$f(x) = \begin{cases} x, & x < 0, \\ 1, & x = 0, \\ x, & 0 < x < 1, \\ 0, & x = 1, \\ x, & x > 1. \end{cases}$$

This is an injective function by the horizontal line test, but is not an increasing function, as $0 < 1$ but $f(0) \not< f(1)$.

10. An (infinite) set $\mathcal{C}$ is said to be *countable* if there exists a bijection between $\mathcal{C}$ and the set $\mathbb{N}$ of natural numbers. This means that the elements of $\mathcal{C}$ can be listed as $c_1, c_2, c_3, \cdots$ so that every element of $\mathcal{C}$ will appear, sooner or later, in the list. For example, $\mathbb{Z}$ is countable, because you can list all the integers in the order $0, 1, -1, 2, -2, 3, -3, \cdots$.
Is the set $\mathbb{Q}$ of all rational numbers countable? How about $\mathbb{R}$ and $\mathbb{C}$?

Solution

$\mathbb{Q}$ is countable. In other words, there is a listing of $\mathbb{Q}$ as (say) $r_1, r_2, r_3, \cdots$ so that every rational number appears in the list. One way to construct such a listing is to plot the rational numbers as points in the upper half-plane: given a fraction $\frac{p}{q}$ in lowest terms with q positive, think of it as located at the point with coordinates (p, q). Then you can start from 0 (located at $(0, 1)$) and trace out a path which spirals outwards, passing through every rational number sooner or later; the rational numbers can be listed in the order in which they occur on this path.

By contrast, $\mathbb{R}$ is not countable. There is a famous proof of this using the expression of real numbers as infinite decimals. Suppose for a contradiction that we had some listing $r_1, r_2, r_3, \cdots$ of the real numbers, so that every real number appeared in the list. We construct an infinite decimal $0.a_1a_2a_3a_4\cdots$ as follows: we choose a_1 to be any digit which is different from the digit which comes first after the decimal point in r_1, a_2 to be any digit which is different from the digit which comes second after the decimal point in r_2, and so on. This new infinite decimal is a real number, so it must be r_i for some i. But this is impossible, since by construction its ith digit after the decimal point is different from that of r_i. This contradiction proves that $\mathbb{R}$ is not countable. [You may have noticed that this proof is not quite right, because some real numbers have more than one decimal expression, e.g. $0.1999\cdots = 0.2000\cdots$. But the proof can be easily modified to take this into account.]

The fact that $\mathbb{R}$ is not countable implies that $\mathbb{C}$ is not countable either, because $\mathbb{R}$ is a subset of $\mathbb{C}$. Indeed, if there were a listing $c_1, c_2, c_3, \cdots$ of $\mathbb{C}$, then we would be able to list the real numbers in the order in which they occur in this bigger list.

Solution to Question 1

(a) The function is neither injective nor surjective (hence not bijective).
(b) Each horizontal line meets the graph of f at exactly one point. Hence f is bijective.
(c) Each horizontal line meets the graph in at least one point; the function is surjective. But the horizontal line $y = 0$ meets the graph at infinitely many points, hence f is not injective.