THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Tutorial Solutions Week 6** 2009

(*These preparatory questions should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

1. Let $f(x) = \lfloor x \rfloor$, the largest integer less than or equal to x. Sketch the graph of f. At which points is f continuous? Right-continuous? Left-continuous?

2. Use the Intermediate Value Theorem to show that there is a solution of the equation $x^2 = \sqrt{x+1}$ in the open interval $(1, 2)$.

Questions for the tutorial

3. Determine whether the functions given by the following formulas are continuous at a.

(a) $h(x) = x^2 + \sqrt{7-x}, \quad a = 4$

(b) $k(x) = \dfrac{x^2-1}{x+1}, \quad a = -1$

(c) $F(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x > 0 \\ 1 - x & \text{if } x \le 0 \end{cases} \quad a = 0$

(d) $K(x) = \begin{cases} \dfrac{x^2-1}{x+1} & \text{if } x \ne -1 \\ 6 & \text{if } x = -1 \end{cases} \quad a = -1$

Solution

(a) The squaring function $x \mapsto x^2$ is continuous everywhere, and the square root function $x \mapsto \sqrt{x}$ is continuous everywhere in its domain $[0, \infty)$, so $h(x)$ is continuous everywhere in its domain $(\infty, 7]$. In particular, it is continuous at 4.

(b) The domain of k does not include -1. Thus the question of whether the function is continuous or discontinuous at -1 is meaningless.

(c) As $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} \dfrac{\sin x}{x} = 1$ (see lecture notes), $\lim_{x\to 0^-} F(x) = \lim_{x\to 0^-} 1 - x = 1$, and $F(0) = 1$, we see that F is continuous at 0.

(d) Since $K(x) = \dfrac{x^2-1}{x+1} = x - 1$ for $x \ne -1$, we have

$$\lim_{x\to -1} K(x) = \lim_{x\to -1} x - 1 = -2.$$

However, $K(-1) = 6$, so $\lim_{x\to -1} K(x) \ne K(-1)$. Therefore K is discontinuous at -1.

4. (a) Find a constant c so that g is continuous everywhere, where g is defined by:

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4, \\ cx + 20 & \text{if } x \ge 4. \end{cases}$$

(b) Repeat part (a) when

$$g(x) = \begin{cases} -c + \sqrt{x-4} & \text{if } x \ge 4, \\ |x^2 - c^2| & \text{if } x < 4. \end{cases}$$

Solution

(a) The functions $x^2 - c^2$ and $cx + 20$, considered on the intervals $(-\infty, 4)$ and $[4, \infty)$ respectively, are continuous for any value of c. Thus the only possible discontinuity is at $x = 4$. For g to be continuous at 4, we require $\lim_{x \to 4^-} g(x) = \lim_{x \to 4^+} g(x) = g(4)$, that is,

$$\lim_{x \to 4^-} (x^2 - c^2) = \lim_{x \to 4^+} (cx + 20) = g(4).$$

Hence $16 - c^2 = 4c + 20$, giving $c = -2$.

(b) As in part (a), for g to be continuous at 4, we require $\lim_{x \to 4^-} g(x) = \lim_{x \to 4^+} g(x) = g(4)$, that is,

$$\lim_{x \to 4^-} |x^2 - c^2| = \lim_{x \to 4^+} (-c + \sqrt{x - 4}) = g(4).$$

Hence we require $|16 - c^2| = -c$. From this we see that $c \leq 0$. If $c \leq -4$ then we require $c^2 - 16 = -c$, that is, $c = \dfrac{-1 - \sqrt{65}}{2} \approx -4.53$. If $-4 < c \leq 0$ then we require $16 - c^2 = -c$, that is, $c = \dfrac{1 - \sqrt{65}}{2} \approx -3.53$. Hence the given function is continuous at 4 for two values of c, namely $c = \dfrac{-\sqrt{65} \pm 1}{2}$.

5. Prove that the equation $\sin x = 2 - x$ has at least one real solution, say α. Use your calculator to find an interval of length 0.01 which contains α.

Solution

Let $f(x) = \sin x - 2 + x$. The function f is continuous on the interval $[0, 2]$, $f(0) = -2$ and $f(2) = \sin 2 \approx 0.91$. Since $-2 < 0 < 0.91$, there is a number c in $(0, 2)$ such that $f(c) = 0$ by the IVT. Thus there is a solution of $\sin x = 2 - x$ in the interval $(0, 2)$.
As $f(1.10) \approx -0.009$ and $f(1.11) \approx 0.006$, there is a solution between 1.10 and 1.11.

6. Use the Intermediate Value Theorem to prove that if $f : [0, 1] \to [0, 1]$ is continuous, there is some c in $[0, 1]$ such that $f(c) = c$.

Solution

If $f(0) = 0$ or $f(1) = 1$ we are done. So suppose that $f(0) \neq 0$, so that $0 < f(0) \leq 1$, and that $f(1) \neq 1$, so that $0 \leq f(1) < 1$. Then if $g(x) = x - f(x)$, g is continuous on $[0, 1]$,

$$g(0) = 0 - f(0) < 0 \quad (\text{as } f(0) > 0)$$

and

$$g(1) = 1 - f(1) > 0 \quad (\text{as } f(1) < 1).$$

By the IVT, there exists a number c in $(0, 1)$ such that $g(c) = 0$, that is, $f(c) = c$.

7. A mountaineer leaves home at 7am and walks to the top of the mountain, arriving at 7pm. The following morning, he starts out at 7am from the top of the mountain and takes the same path back, arriving home at 7pm. Use the IVT to show that there is a point on the path that he will cross at exactly the same time of day on both days.

Solution

Define $u(t)$ to be the mountaineer's distance from home, as a function of time (measured in hours), on the first day. Likewise, define $d(t)$ to be the mountaineer's distance from home, as a function of time, on the second day. Let D be the distance from home to the top of the mountain. We know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$.
We may assume that u and d are continuous functions on the interval $[0, 12]$. Now consider the function $u - d$ which is also continuous on $[0, 12]$. We have $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the IVT, there is some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0$, or equivalently, $u(t_0) = d(t_0)$. Thus, at the same time t_0 after 7am, the mountaineer will be at the same place on both days.

8. (a) Show that if f is continuous on the closed interval $[a, b]$, then there is a function g which is continuous on $\mathbb{R}$, and which satisfies $g(x) = f(x)$ for all x in $[a, b]$.

(b) Give an example to show that the previous assertion is false if $[a, b]$ is replaced by the open interval (a, b).

Solution

(a) There are many possibilities for g. A simple example is the function defined as follows.

$$g(x) = \begin{cases} f(a) & \text{if} \quad x \in (-\infty, a) \\ f(x) & \text{if} \quad x \in [a, b] \\ f(b) & \text{if} \quad x \in (b, \infty) \end{cases}$$

Then g is continuous on $\mathbb{R}$. To see this, observe that the only possible points of discontinuity are at a and b. At a we have $\lim_{x \to a^-} g(x) = f(a) = \lim_{x \to a^+} g(x)$. A similar statement shows g is continuous at b.

(b) The function f defined by $f(x) = \dfrac{1}{x}$ is continuous on the open interval $(0, 1)$ and as $\lim_{x \to 0^+} f(x) = \infty$, it is not possible to define g so that $g(0) = \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} f(x)$.

9. Prove that if f is continuous at a, then $|f|$ is continuous at a. Is the converse true?

Solution

Given any $\epsilon > 0$, we know there exists $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

But for any real numbers r, s it is true that $|r - s| \geq |r| - |s|$ and also that $|r - s| \geq |s| - |r|$. Hence for all r, s, we have $\big|\,|r| - |s|\,\big| \leq |r - s|$. This shows that

$$\big|\,|f(x)| - |f(a)|\,\big| \leq |f(x) - f(a)|.$$

We are therefore guaranteed that $\big|\,|f(x)| - |f(a)|\,\big| < \epsilon$ whenever $|f(x) - f(a)| < \epsilon$, which occurs for all x satisfying $|x - a| < \delta$. Thus $|f|$ is continuous at a. The converse assertion – that if $|f|$ is continuous at a, then so is f – is false. For instance, let f be the function defined by

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Then $|f|$ is the constant function with value 1, so it is continuous at 0, but clearly f is not continuous at 0.

Extra Questions

10. Prove the substitution law for limits. That is, assuming that $\lim_{x \to a} g(x) = b$ and that f is continuous at b, prove (using the ϵ, δ definition) that $\lim_{x \to a} f(g(x)) = f(b)$.

Solution

Since f is continuous at b, we have $\lim_{y \to b} f(y) = f(b)$. Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that

$$|y - b| < \delta_1 \implies |f(y) - f(b)| < \epsilon. \qquad (1)$$

In particular, this means that

$$|g(x) - b| < \delta_1 \implies |f(g(x)) - f(b)| < \epsilon. \qquad (2)$$

Since $\delta_1 > 0$ and $\lim_{x \to a} g(x) = b$, there exists $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - b| < \delta_1. \qquad (3)$$

From (2) and (3) we see that

$$0 < |x - a| < \delta_2 \implies |f(g(x)) - f(b)| < \epsilon,$$

that is, $\lim_{x \to a} f(g(x))$ exists and equals $f(b)$.

11. Suppose that f is continuous on the closed interval $[a, b]$. A theorem mentioned in lectures says that the set of values $\{f(x) \,|\, a \leq x \leq b\}$ has an upper bound; therefore it has a least upper bound, say α. Assuming this, prove that there is some $c \in [a, b]$ such that $f(c) = \alpha$, and deduce the Extreme Value Theorem. (*Hint*: assume that no such c exists and derive a contradiction by considering the function $g(x) = \dfrac{1}{\alpha - f(x)}$.)

Solution

As the hint suggests, we proceed by contradiction. Assume that there is no $c \in [a, b]$ such that $f(c) = \alpha$. Then $g(x) = \dfrac{1}{\alpha - f(x)}$ is a well-defined continuous function on the whole interval $[a, b]$, which takes positive values since α is an upper bound for the values of f. By the same theorem that the question invoked for f, the set of values $\{g(x) \,|\, a \leq x \leq b\}$ has an upper bound, say β (some positive number). This means that

$$\frac{1}{\alpha - f(x)} \leq \beta \ \text{ for all } \ x \in [a, b].$$

Hence $\alpha - f(x) \geq \dfrac{1}{\beta}$ for all $x \in [a, b]$, which rearranges to $f(x) \leq \alpha - \dfrac{1}{\beta}$ for all $x \in [a, b]$. But this says that $\alpha - \dfrac{1}{\beta}$ is an upper bound for $\{f(x) \,|\, a \leq x \leq b\}$, which is impossible because α was chosen to be the least upper bound. This contradiction means that the original assumption must have been wrong, so there is some $c \in [a, b]$ such that $f(c) = \alpha$. We then have $f(x) \leq f(c)$ for all $x \in [a, b]$. Applying the same argument to $-f$ in place of f, we see that there must also be some $d \in [a, b]$ such that $f(x) \geq f(d)$ for all $x \in [a, b]$. So we have proved the Extreme Value Theorem.

12. Consider the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \dfrac{1}{q} & \text{if } x = \dfrac{p}{q},\ q > 0, \text{ for } p \text{ and } q \text{ relatively prime integers.} \\ & \text{(That is, } \dfrac{p}{q} \text{ is the expression of } x \text{ in lowest terms.)} \end{cases}$$

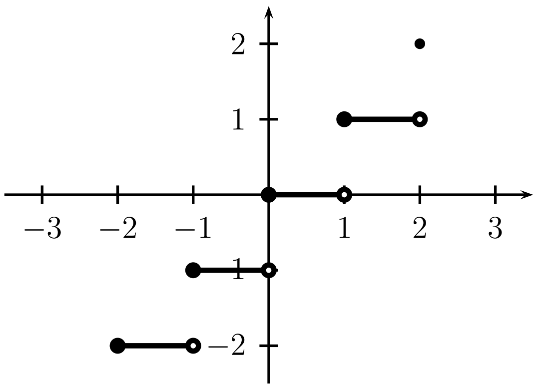
(In particular, $f(0) = 1$ since 0 in lowest terms is $\frac{0}{1}$.) Prove that f is discontinuous at 0, and at every other rational number.

Solution

Let a be any rational number, and suppose (for a contradiction) that f is continuous at a. We have $f(a) > 0$, so by a result proved in lectures there is a $\delta > 0$ such that $f(x) > 0$ for all x satisfying $|x - a| < \delta$. But then if we let y be any irrational number such that $0 < y < \delta$, $x = a + y$ satisfies this condition and yet is irrational, so $f(x) = 0$. This is the desired contradiction, proving that f is discontinuous at every rational number (in particular, it has infinitely many discontinuities).
It turns out that $\lim_{x \to a} f(x) = 0$ for all $a \in \mathbb{R}$. (So f is continuous at a for every irrational number a.) Can you prove this?

Solution to Question 1

We can write $f(x) = n$ for $x \in [n, n+1), n \in \mathbb{Z}$. The graph is a step function.



Since f is a constant function on each open interval $(n, n+1)$, it is continuous at all points which are not integers. At any integer point n, we have $\lim_{x \to n^+} f(x) = n = f(n)$, so f is also right-continuous at all integer points. Note that f is not left-continuous at integer points n since $\lim_{x \to n^-} f(x) = n - 1 \neq f(n)$.

Solution to Question 2

Let $f(x) = x^2 - \sqrt{x+1}$. The function f is continuous on the interval $[1, 2]$, $f(1) = 1 - \sqrt{2}$ and $f(2) = 4 - \sqrt{3}$. Since $1 - \sqrt{2} < 0 < 4 - \sqrt{3}$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the IVT. Thus there is a solution of $x^2 = \sqrt{x+1}$ in the interval $(1, 2)$.