THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Tutorial Solutions Week 8** 2009

(*These preparatory questions should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

1. The function f is defined by $f(x) = |x-1|$. Sketch its graph. Observe that there is no value c such that $f(3) - f(0) = f'(c)(3-0)$. Why does this not contradict the Mean Value Theorem?

2. Using l'Hôpital's rule, find $\displaystyle\lim_{x\to\frac{3\pi}{2}} \frac{\cos x}{x-(3\pi/2)}$.

Questions for the tutorial

3. Use the Mean Value Theorem to prove the following inequalities:

(a) $|\cos y - \cos x| \le |y-x|$, for all real numbers x and y;

(b) $|\sinh x| \ge |x|$ for all real x;

(c) $e^x \ge 1 + x$ for all real x.

Solution

(a) It suffices to prove this when $x < y$. The cosine function is certainly continuous and differentiable everywhere, so we can apply the Mean Value Theorem to the interval $[x, y]$. It says that there is a number c, $x < c < y$, such that

$$\frac{\cos y - \cos x}{y-x} = -\sin c.$$

Since $-1 \le -\sin c \le 1$, we have

$$-1 \le \frac{\cos y - \cos x}{y-x} \le 1,$$

and so

$$|\cos y - \cos x| \le y - x.$$

(b) When $x = 0$, both sides of the inequality are 0. Since $\sinh(-x) = -\sinh x$, it suffices to prove the inequality when $x > 0$. Apply the MVT to $f(t) = \sinh t$ on the interval $[0, x]$. (Note that f is continuous and differentiable everywhere so the hypotheses of the MVT hold.) There exists $c \in (0, x)$ such that

$$\frac{\sinh x - \sinh 0}{x-0} = f'(c) = \cosh c > 1.$$

That is, $\frac{|\sinh x|}{|x|} > 1$, so $|\sinh x| > |x|$ as required.

(c) When $x = 0$, both sides of the inequality are 1. Suppose that $x < 0$. Apply the MVT to $f(t) = e^t$ on the interval $[x, 0]$. (Note that f is continuous and differentiable everywhere so the hypotheses of the MVT hold.) There exists $c \in (x, 0)$ such that

$$\frac{e^0 - e^x}{0-x} = f'(c) = e^c < 1.$$

That is, $1 - e^x < -x$, so $e^x > 1 + x$.
If $x > 0$, apply the MVT to $f(t) = e^t$ on the interval $[0, x]$. There exists $c \in (0, x)$ such that

$$\frac{e^x - e^0}{x - 0} = f'(c) = e^c > 1.$$

That is, $e^x - 1 > x$, so $e^x > 1 = x$ as before. We conclude that for all x, $e^x \geq 1 + x$.

4. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. Use the Mean Value Theorem to show that the largest possible value of $f(2)$ is 7.

Solution

We are given that f is differentiable on $\mathbb{R}$, and therefore f is continuous on $\mathbb{R}$. In particular, f is continuous on the closed interval $[0, 2]$ and differentiable on the open interval $(0, 2)$, as is required for application of the Mean Value Theorem for $f(x)$ on the interval $[0, 2]$. Applying the MVT to $[0, 2]$, we see that there is a number c in $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{f(2) + 3}{2}.$$

Therefore $f(2) = 2f'(c) - 3 \leq 2 \times 5 - 3 = 7$. The largest possible value of $f(2)$ is 7.

5. The road between two towns, A and B, is 110 km long. You left A to drive to B at the same time as I left B to drive to A. We met exactly 30 minutes later. Use the Mean Value Theorem to prove that at least one of us exceeded the speed limit, 100 km/h, by at least 10 km/h.

Solution

Let $f(t)$ km be the distance that you have travelled and $g(t)$ km the distance I have travelled t hours after departure from our starting points. Certainly $f(0.5) + g(0.5) = 110$, so either $f(0.5)$ or $g(0.5)$ is at least 55. We take f and g to be continuous on $[0, 0.5]$ and differentiable on $(0, 0.5)$ so the hypotheses of the Mean Value Theorem hold. Then

$$f'(c_1) = \frac{f(0.5) - f(0)}{0.5 - 0} = 2f(0.5)$$

and

$$g'(c_2) = \frac{g(0.5) - g(0)}{0.5 - 0} = 2g(0.5)$$

for some numbers $c_1, c_2 \in (0, 0.5)$. Hence either $f'(c_1) \geq 2 \times 55 = 110$ or $g'(c_2) \geq 2 \times 55 = 110$, which means that one (or both) of us travelled at a speed of at least 110 km/hour at some moment before we met.

6. Find the following limits.

(a) $\displaystyle\lim_{x \to -1} \frac{x^6 - 1}{x^4 - 1}$ (b) $\displaystyle\lim_{x \to \pi} \frac{\tan x}{x}$ (c) $\displaystyle\lim_{x \to \infty} \frac{\ln x}{\ln(\ln x)}$

(d) $\displaystyle\lim_{x \to \infty} \frac{\ln x}{x^{1/100}}$ (e) $\displaystyle\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$ (f) $\displaystyle\lim_{x \to -\infty} \frac{x}{\sqrt{x^2 + 1}}$

Solution

(a) Let $f(x) = x^6 - 1$ and $g(x) = x^4 - 1$. Then $\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = 0$. Also, $f'(x) = 6x^5$ and $g'(x) = 4x^3$ and $\lim_{x\to -1} \frac{f'(x)}{g'(x)} = \lim_{x\to -1} \frac{6x^5}{4x^3} = \frac{3}{2}$. Thus, using l'Hôpital's rule,

$$\lim_{x\to -1} \frac{x^6 - 1}{x^4 - 1} = \frac{3}{2}.$$

(b) $\lim_{x\to\pi} \tan x = 0$ and $\lim_{x\to\pi} x = \pi \neq 0$. It follows that

$$\lim_{x\to\pi} \frac{\tan x}{x} = \frac{\lim_{x\to\pi} \tan x}{\lim_{x\to\pi} x} = \frac{0}{\pi} = 0.$$

(c) In anticipation of a successful application of l'Hôpital's rule, we often write our solutions in a more informal manner, as in this part and the following parts. Here we have $\lim_{x\to\infty} \frac{\ln x}{\ln(\ln x)} = \lim_{x\to\infty} \frac{1/x}{1/(x \ln x)} = \lim_{x\to\infty} \ln x = \infty$.

(d) Using l'Hôpital's rule, $\lim_{x\to\infty} \frac{\ln x}{x^{1/100}} = \lim_{x\to\infty} \frac{1/x}{(1/100)x^{-99/100}} = \lim_{x\to\infty} \frac{100}{x^{1/100}} = 0$.

(e) Using l'Hôpital's rule twice, $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$.

(f) Using l'Hôpital's rule twice we see that we end up with the same limit, so that the rule cannot be applied successfully. Fortunately, we can evaluate this limit directly, as

$$\lim_{x\to -\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to -\infty} -\sqrt{\frac{x^2}{x^2+1}} = \lim_{x\to -\infty} -\sqrt{\frac{1}{1+\frac{1}{x^2}}} = -1.$$

7. Use induction on n and l'Hôpital's rule to prove that $\lim_{x\to 0^+} x(\ln x)^n = 0$ for $n = 0, 1, 2, \cdots$.

Solution

For $n = 0$, $\lim_{x\to 0^+} x(\ln x)^n = \lim_{x\to 0^+} x = 0$.
Assume that $\lim_{x\to 0^+} x(\ln x)^n = 0$ (induction hypothesis). Now

$$\begin{aligned} \lim_{x\to 0^+} x(\ln x)^{n+1} &= \lim_{x\to 0^+} \frac{(\ln x)^{n+1}}{1/x} && \text{(of the form } \pm \tfrac{\infty}{\infty}\text{)} \\ &= \lim_{x\to 0^+} \frac{(n+1)(\ln x)^n (1/x)}{-1/x^2} && \text{(by l'Hôpital's rule)} \\ &= \lim_{x\to 0^+} -(n+1)x(\ln x)^n, \end{aligned}$$

and this is equal to 0 by the induction hypothesis. So the result is true by induction.

8. Find the following limits.

(a) $\lim_{x\to\infty} x^{1/x}$ (*Hint*: Set $y = x^{1/x}$, and compute $\lim_{x\to\infty} \ln y$.)

(b) $\lim_{x\to 0^+} x^{1/x}$

(c) $\lim_{x\to\infty} (1 + e^{-x})^x$

(d) $\lim_{x\to \frac{\pi}{2}^-} (\tan x)^{\cos x}$

(e) $\lim_{x\to 0^+} (\sinh \frac{4}{x})^x$

Solution

(a) Let $y = x^{1/x}$. Then $\ln y = \dfrac{\ln x}{x}$ and $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \dfrac{\ln x}{x} = \lim_{x\to\infty} \dfrac{1/x}{1} = 0$.

Therefore

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e^0 = 1.$$

(b) Let $y = x^{1/x}$. Then $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \dfrac{\ln x}{x} = -\infty$. (Note: The last limit is of the form $\dfrac{-\infty}{0}$ and l'Hôpital's rule does not apply.) Now, $\ln y \to -\infty$ implies $y \to 0$. That is $\lim_{x\to 0^+} x^{1/x} = 0$.

(c) Let $y = (1+e^{-x})^x$. Then $\ln y = \dfrac{\ln(1+e^{-x})}{1/x}$ and

$$\begin{aligned}
\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} \frac{\ln(1+e^{-x})}{1/x} \quad (\frac{0}{0} \text{ form}) \\
&= \lim_{x\to\infty} \frac{x^2 e^{-x}}{1+e^{-x}} \\
&= \lim_{x\to\infty} \frac{x^2}{e^x+1} \quad (\frac{\infty}{\infty} \text{ form}) \\
&= \lim_{x\to\infty} \frac{2x}{e^x} \\
&= \lim_{x\to\infty} \frac{2}{e^x} \\
&= 0.
\end{aligned}$$

It follows that $y \to 1$. Hence $\lim_{x\to\infty} (1+e^{-x})^x = 1$.

(d) We write $(\tan x)^{\cos x}$ as $e^{\cos x \ln(\tan x)}$ and evaluate $\lim_{x\to \frac{\pi}{2}^-} \cos x \ln(\tan x)$. Now

$$\lim_{x\to \frac{\pi}{2}^-} \cos x \ln(\tan x) = \lim_{x\to \frac{\pi}{2}^-} \sin x \frac{\ln(\tan x)}{\tan x}.$$

But $\lim_{x\to \frac{\pi}{2}^-} \dfrac{\ln(\tan x)}{\tan x} = 0$, using l'Hôpital's rule. As $\lim_{x\to \frac{\pi}{2}^-} \sin x = 1$, we see that $\lim_{x\to \frac{\pi}{2}^-} \cos x \ln(\tan x) = 0$. As the exponential function is continuous, the Substitution Law then gives the final result,

$$\lim_{x\to \frac{\pi}{2}^-} (\tan x)^{\cos x} = e^0 = 1.$$

(e) Let $y = (\sinh \frac{4}{x})^x$. Then $\ln y = \dfrac{\ln(\sinh \frac{4}{x})}{1/x}$ and

$$\begin{aligned}
\lim_{x\to 0^+} \ln y &= \lim_{x\to 0^+} \frac{\ln(\sinh \frac{4}{x})}{1/x} \qquad (\tfrac{\infty}{\infty} \text{ form})\\
&= \lim_{x\to 0^+} \frac{\frac{-\frac{4}{x^2}\cosh\frac{4}{x}}{\sinh\frac{4}{x}}}{\frac{-1}{x^2}}\\
&= 4 \lim_{x\to 0^+} \frac{\cosh\frac{4}{x}}{\sinh\frac{4}{x}}\\
&= 4 \lim_{x\to 0^+} \left(\frac{e^{4/x}+e^{-4/x}}{e^{4/x}-e^{-4/x}}\right)\\
&= 4 \lim_{x\to 0^+} \left(\frac{1+e^{-8/x}}{1-e^{-8/x}}\right)\\
&= 4.
\end{aligned}$$

Hence $\lim_{x\to 0^+} (\sinh \frac{4}{x})^x = \lim_{x\to 0^+} e^{\ln y} = e^4$.

Extra Questions

9. What is wrong with the following "proof" of the Cauchy Mean Value Theorem?

CMVT: If f and g are continuous on $[a,b]$ and differentiable on (a,b), then there is a number x in (a,b) such that $(f(b)-f(a))g'(x) = (g(b)-g(a))f'(x)$.

"Proof": Applying the Mean Value Theorem to f and g separately, we find that there is an x in (a,b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(x) \quad \text{and} \quad \frac{g(b)-g(a)}{b-a} = g'(x).$$

Therefore

$$(f(b)-f(a))g'(x) = (b-a)f'(x)g'(x) = (g(b)-g(a))f'(x),$$

which proves the Theorem.

Solution

We can be sure that there is an x in (a,b) such that $\frac{f(b)-f(a)}{b-a} = f'(x)$ and that there is a y in (a,b) such that $\frac{g(b)-g(a)}{b-a} = g'(y)$, but there is no guarantee that x and y are the same number.

10. Consider the statement: "if f and g are differentiable, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} \dfrac{f(x)}{g(x)} = L$, then $\lim_{x\to a} \dfrac{f'(x)}{g'(x)} = L$". Show that this is false by giving a counter-example.

Solution

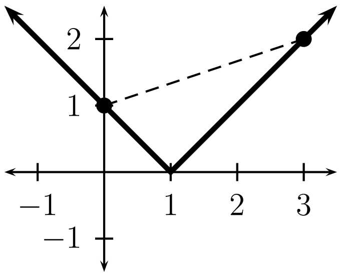
One counter-example is $a = 0$, $g(x) = x$, and f defined by $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$ We saw in the previous Tutorial that f is differentiable everywhere, which in particular means that f is continuous at 0, so $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$. We also have

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x\to 0} x \sin \frac{1}{x} = 0,$$

by the squeeze law. However, we saw in the previous Tutorial that $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} f'(x)$ does not exist, so the conclusion of the above statement does not hold.

Solution to Question 1

The graph of f is as follows.



Now $f(3) - f(0) = 1$, and there is no value c such that $1 = 3f'(c)$. This does not contradict the Mean Value Theorem because f is not differentiable on the interval $(0, 3)$. (Specifically, $f'(1)$ does not exist.)

Solution to Question 2

Let $f(x) = \cos x$ and $g(x) = x - \frac{3\pi}{2}$. Then $\lim_{x\to \frac{3\pi}{2}} f(x) = \lim_{x\to \frac{3\pi}{2}} g(x) = 0$, and

$$\lim_{x\to \frac{3\pi}{2}} \frac{f'(x)}{g'(x)} = \lim_{x\to \frac{3\pi}{2}} (-\sin x) = -\sin \frac{3\pi}{2} = 1.$$

So l'Hôpital's rule says that $\lim_{x\to \frac{3\pi}{2}} \dfrac{\cos x}{x - (3\pi/2)} = 1$.