THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1 **Tutorial Solutions Week 11** 2009

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Compute the partial derivatives $f_x(x, y)$, $f_y(x, y)$ of the following functions $f(x, y)$.

(a) xy^3 (b) $\sin(2x + 3y)$ (c) $\ln(x + \sqrt{x^2 + y^2})$

Questions for the tutorial

2. Find the limit, if it exists, or show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(2,3)} (x^2y^2 - 2xy^5 + 3y)$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^3 + x^3y^2 - 5}{2 - xy}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x - y}{x^2 + y^2}$ (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + xy^2}{x^2 + y^2}$

Solution

(a) The function is a polynomial, so the limit equals $(2^2)(3^2) - 2(2)(3^5) + 3(3) = -927$.

(b) Since this is a rational function defined at $(0, 0)$, the limit equals $(0 + 0 - 5)/(2 - 0) = -\frac{5}{2}$.

(c) Let $f(x, y) = (x - y)/(x^2 + y^2)$.
Approach $(0, 0)$ along the x-axis. Let $x = t$, $y = 0$. As $(x, y) \to (0, 0)$, we have $t \to 0$.
Then $f(t, 0) = t/t^2 = 1/t$, $\displaystyle\lim_{t\to0^+} f(t, 0) = \infty$ and $\displaystyle\lim_{t\to0^-} f(t, 0) = -\infty$.
Thus $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ doesn't exist.

(d) $\displaystyle\lim_{(x,y)\to(0,0)} (x^3 + xy^2)/(x^2 + y^2) = \lim_{(x,y)\to(0,0)} x(x^2 + y^2)/(x^2 + y^2) = \lim_{(x,y)\to(0,0)} x = 0.$

3. Consider the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}, \text{ defined for } (x, y) \neq (0, 0).$$

Is it possible to define $f(0, 0)$ so that f is continuous at $(0, 0)$?

Solution

Using polar coordinates for x and y, (that is, $x = r\cos\theta$, $y = r\sin\theta$), we have

$$\frac{\sin(x^2 + y^2)}{x^2 + y^2} = \frac{\sin r^2}{r^2}.$$

Since $(x, y) \to (0, 0)$ if and only if $r^2 \to 0$, we see that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r^2\to0} \frac{\sin r^2}{r^2} = 1.$$

Thus we can define $f(0, 0) = 1$ to make f continuous at $(0, 0)$.

4. Decide whether the limits exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}\sin\frac{1}{x^2+y^4}$

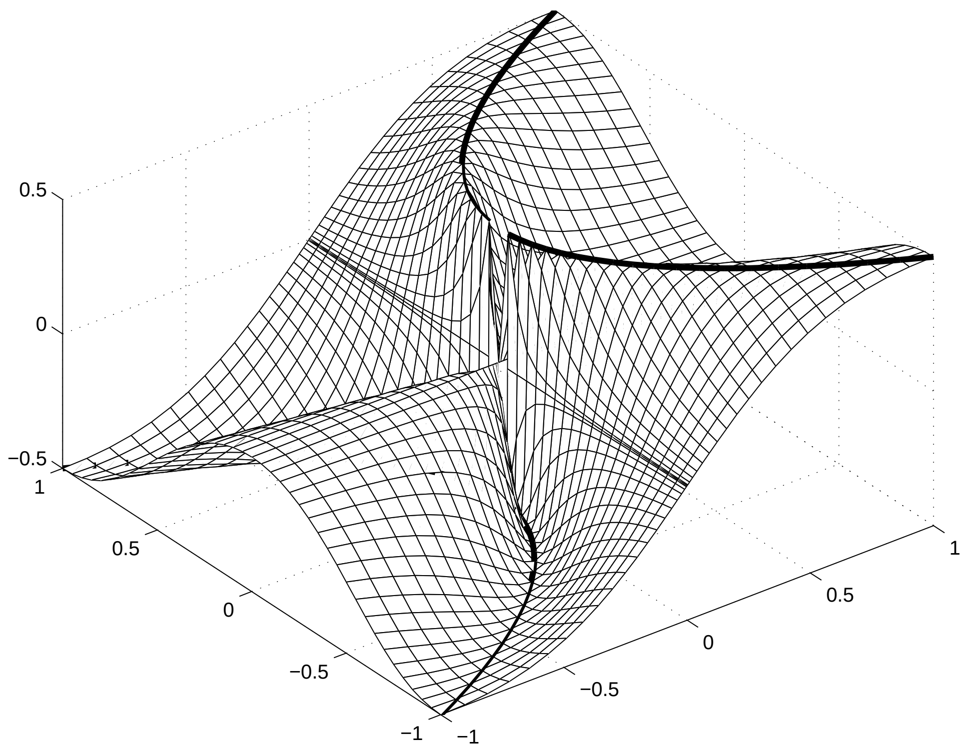
(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$

Solution

(a) Let $u = y^2$. Then $(x, y) \to (0, 0)$ if and only if $(x, u) \to (0, 0)$. We now use polar coordinates $x = r\cos\theta$, $u = r\sin\theta$ to show that the limit does not exist. We have

$$\frac{xy^2}{x^2+y^4} = \frac{xu}{x^2+u^2} = \frac{r^2\cos\theta\sin\theta}{r^2} = \frac{\sin 2\theta}{2}.$$

If $(x, u) \to (0, 0)$ along the positive x axis (where $\theta = 0$), the limit is 0; if $(x, u) \to (0, 0)$ along the line $u = x$ in the first quadrant (where $\theta = \pi/4$), the limit is $1/2$. Hence $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist. Here is an image of the surface drawn using the original variables x, y, in which one particular path to the origin has been highlighted. It's a path along the parabola $x = y^2$, where the limit is $\frac{1}{2}$.



(b) The sine function is bounded between -1 and 1. It is easy to show, using polar coordinates, that $\displaystyle\lim \frac{xy^2}{x^2+y^2} = 0$. Hence the limit exists and equals 0.

(c) let $f(x, y) = \dfrac{x^2-y^2}{x^2+y^2}$. Suppose that (x, y) approaches $(0, 0)$ along the x axis. Then $(x, y) = (t, 0)$ and

$$\lim_{t\to 0} f(t, 0) = \lim_{t\to 0} \frac{t^2}{t^2} = 1.$$

However, if (x, y) approaches $(0, 0)$ along the y axis, then $(x, y) = (0, t)$ and

$$\lim_{t\to 0} f(0, t) = \lim_{t\to 0} \frac{-t^2}{t^2} = -1.$$

Hence no limit exists.

(d) Using polar coordinates, we see that

$$\frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \frac{r^2(\cos^2\theta - \sin^2\theta)}{r} = r\cos 2\theta.$$

As $-r \le r\cos 2\theta \le r$, we see that $\lim_{r\to 0} r\cos 2\theta = 0$, by the Squeeze Law. Hence $\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = 0$.

5. Define $f : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$f(x,y) = \begin{cases} 1 & \text{if } x = y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is not continuous at $(0,0)$ but both f_x and f_y exist at $(0,0)$.

Solution

If (x,y) approaches $(0,0)$ along the line $y = x$, then $\lim_{(x,y)\to(0,0)} f(x,y) = 1 \neq f(0,0)$. Therefore f is not continuous at $(0,0)$. However both partial derivatives exist at the origin:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \frac{0-0}{h} = 0,$$

and

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \frac{0-0}{k} = 0.$$

6. Verify that the functions given by the following formulas are solutions of the *Laplace equation* $f_{xx} + f_{yy} = 0$.

(a) $x^2 - y^2$ (b) $2xy$ (c) $e^x \cos y$ (d) $e^x \sin y$

Solution

(a) $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = -2$, so their sum is zero, as required.

(b) Both $f_{xx}(x,y)$ and $f_{yy}(x,y)$ are zero.

(c) $f_{xx}(x,y) = e^x \cos y$, $f_{yy}(x,y) = -e^x \cos y$, so their sum is zero.

(d) $f_{xx}(x,y) = e^x \sin y$, $f_{yy}(x,y) = -e^x \sin y$, so their sum is zero.

7. Suppose that f is a diffentiable function of one variable. Show that if $z = f\left(\frac{x}{y}\right)$, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

Solution

Differentiating z with respect to x (holding y constant) gives

$$\frac{\partial z}{\partial x} = \frac{1}{y} f'\left(\frac{x}{y}\right)$$

and differentiating z with respect to y (holding x constant) gives

$$\frac{\partial z}{\partial y} = -\frac{x}{y^2} f'\left(\frac{x}{y}\right).$$

We then obtain

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x}{y} f'\left(\frac{x}{y}\right) - \frac{xy}{y^2} f'\left(\frac{x}{y}\right) = \frac{x}{y} f'\left(\frac{x}{y}\right) - \frac{x}{y} f'\left(\frac{x}{y}\right) = 0.$$

8. Find the equation of the tangent plane to the surface $z = e^x \ln y$ at $(3, 1, 0)$.

Solution

Put $f(x,y) = e^x \ln y$. Then $f_x(x,y) = e^x \ln y$ and $f_y(x,y) = \dfrac{e^x}{y}$. So $f_x(3,1) = 0$ and $f_y(3,1) = e^3$. Thus the equation of the tangent plane is

$$z - 0 = 0(x-3) + e^3(y-1),$$

that is, $z = e^3 y - e^3$.

9. Find the single point at which the tangent plane to the surface $z = x^2 + 2xy + 2y^2 - 6x + 8y$ is horizontal.

Solution

At the point corresponding to $x = a$, $y = b$, the tangent plane has equation

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

This is horizontal (that is, it's of the form $z =$ constant) when $f_x(a,b) = f_y(a,b) = 0$. Now $f_x(a,b) = 2a + 2b - 6$ and$f_y(a,b) = 2a + 4b + 8$. Setting each expression equal to 0 and solving simultaneously gives $a = 10$, $b = -7$. The required point on the surface is then $(10, -7, -58)$.

Extra Question

10. Use the ϵ, δ definition of the limit of a function of two variables to show that

$$\lim_{(x,y)\to(1,2)} x^2 + y = 3.$$

Solution

We want to show that given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |(x,y) - (1,2)| < \delta \implies |x^2 + y - 3| < \epsilon.$$

Note that the set of points (x,y) satisfying the inequality $0 < |(x,y) - (1,2)| < \delta$ can be interpreted geometrically as the set of points in the interior of a circle with centre $(1,2)$ and radius δ, without the centre itself.

We examine the difference between $x^2 + y$ and 3 and try to write this in such a way as to incorporate terms in $x - 1$ and $y - 2$.

$$\begin{aligned}|x^2 + y - 3| &= |(x-1)^2 + 2x - 1 + (y-2) + 2 - 3| \\ &= |(x-1)^2 + 2(x-1) + (y-2)| \\ &\le (x-1)^2 + 2|x-1| + |y-2|\end{aligned}$$

To guarantee that $|x^2 + y - 3| < \epsilon$, we need only be sure that each of the three expressions $(x-1)^2$, $2|x-1|$, $|y-2|$ is less than $\epsilon/3$. Now as $\lim_{x\to 1}(x-1)^2 = 0$, $\lim_{x\to 1} 2|x-1| = 0$ and $\lim_{y\to 2}|y-2| = 0$, there exists $\delta_1 > 0$ such that

$$0 < |x-1| < \delta_1 \implies (x-1)^2 < \epsilon/3$$

(for example, $\delta_1 = \sqrt{\epsilon/3}$), there exists $\delta_2 > 0$ such that

$$0 < |x-1| < \delta_2 \implies 2|x-1| < \epsilon/3$$

($\delta_2 = \epsilon/6$), and there exists $\delta_3 > 0$ such that

$$0 < |y-2| < \delta_3 \implies |y-2| < \epsilon/3$$

($\delta_3 = \epsilon/3$). Now choose δ to be the minimum of δ_1, δ_2, δ_3. Then whenever (x, y) is a point inside a circle with centre at $(1, 2)$ and radius δ (but not the centre itself), we can be sure that $|x^2 + y - 3| < \epsilon$. That is, given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |(x, y) - (1, 2)| < \delta \implies |x^2 + y - 3| < \epsilon.$$

This proves the result.

Solution to Question 1

(a) $f_x = y^3$, $f_y = 3xy^2$

(b) $f_x = 2\cos(2x + 3y)$, $f_y = 3\cos(2x + 3y)$

(c) $f_x = \dfrac{1 + x(x^2+y^2)^{-1/2}}{x + \sqrt{x^2+y^2}} = \dfrac{1}{\sqrt{x^2+y^2}}$, $f_y = \dfrac{y}{(x + \sqrt{x^2+y^2})\sqrt{x^2+y^2}}$