Semester 1 **Tutorial Solutions Week 13** 2009

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Find the directional derivative of $f(x,y) = x^2 + 2e^{x+y}$ in the direction of $\mathbf{v} = \mathbf{i} - \mathbf{j}$ at the point $(1,2)$.

Questions for the tutorial

2. Use the formula $\dfrac{dy}{dx} = -\dfrac{f_x(x,y)}{f_y(x,y)}$ to find an expression for $\dfrac{dy}{dx}$ where y is defined implicitly as a function of x by the equation $x^3 + y^3 = 3xy$. Hence evaluate the slope of the tangent to the curve $x^3 + y^3 = 3xy$ at the point $(2/3, 4/3)$.

Solution

Put $f(x,y) = x^3 + y^3 - 3xy$, so that $f(x,y) = 0$ is the equation of the curve.
As $\dfrac{\partial f}{\partial x} = 3x^2 - 3y$ and $\dfrac{\partial f}{\partial y} = 3y^2 - 3x$, we have

$$\frac{dy}{dx} = -\frac{f_x(x,y)}{f_y(x,y)} = -\frac{3x^2 - 3y}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}.$$

At the point $(2/3, 4/3)$, the slope of the tangent to the curve is

$$\frac{4/3 - 4/9}{16/9 - 2/3} = 4/5.$$

3. Let $f(x,y) = 1 + 2x\sqrt{y}$ and $g(x,y) = e^{-x}\sin y$.

(a) Find $\nabla f(x,y)$, $\nabla f(3,4)$, $\nabla g(x,y)$, $\nabla g(2,0)$.

(b) Let $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$. Determine the unit vector $\widehat{\mathbf{v}}$. Hence find $D_{\widehat{\mathbf{v}}}f(x,y)$ and also the special case $D_{\widehat{\mathbf{v}}}f(3,4)$. Similarly, if $\mathbf{w} = 3\mathbf{i} + 2\mathbf{j}$, find $D_{\widehat{\mathbf{w}}}g(x,y)$ and $D_{\widehat{\mathbf{w}}}g(2,0)$.

Solution

(a) $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = 2\sqrt{y}\,\mathbf{i} + \dfrac{x}{\sqrt{y}}\mathbf{j}$, $\nabla f(3,4) = 4\,\mathbf{i} + \dfrac{3}{2}\mathbf{j}$,

$\nabla g(x,y) = -e^{-x}\sin y\,\mathbf{i} + e^{-x}\cos y\,\mathbf{j}$, $\nabla g(2,0) = e^{-2}\,\mathbf{j}$.

(b) The unit vector $\widehat{\mathbf{v}}$ in the direction of $\mathbf{v}$ is given by $\widehat{\mathbf{v}} = \dfrac{4}{5}\mathbf{i} - \dfrac{3}{5}\mathbf{j}$. Therefore $D_{\widehat{\mathbf{v}}}f(x,y) = \dfrac{8}{5}\sqrt{y} - \dfrac{3x}{5\sqrt{y}}$ and $D_{\widehat{\mathbf{v}}}f(3,4) = \dfrac{16}{5} - \dfrac{9}{10} = \dfrac{23}{10}$.

The unit vector $\widehat{\mathbf{w}} = \dfrac{3}{\sqrt{13}}\mathbf{i} + \dfrac{2}{\sqrt{13}}\mathbf{j}$, so that $D_{\widehat{\mathbf{w}}}g(x,y) = \dfrac{e^{-x}}{\sqrt{13}}(-3\sin y + 2\cos y)$ and $D_{\widehat{\mathbf{w}}}(2,0) = \dfrac{2e^{-2}}{\sqrt{13}}$.

4. Instead of the one-sided limit used in the definition of the directional derivative in this course, many texts use the following two-sided limit:

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector and h may be either positive or negative.

(a) Let $f(x, y) = \sqrt{xy}$ and let $\mathbf{u}$ be a unit vector. Prove that $D_{\mathbf{u}}f(0, 0)$, defined using the two-sided limit above, exists if and only if $\mathbf{u} = \mathbf{i}$, $-\mathbf{i}$, $\mathbf{j}$ or $-\mathbf{j}$.

(b) Now use our one-sided definition for the limit and find all directions for which $D_{\mathbf{u}}f(0, 0)$ exists.

Solution

(a) The domain of f is $\{ (x, y) \mid x, y \geq 0 \text{ or } x, y \leq 0 \}$, that is, the 1st and 3rd quadrants of the xy-plane including the axes. By definition,

$$D_{\mathbf{u}}f(0, 0) = \lim_{h \to 0} \frac{f(0 + hu_1, 0 + hu_2) - f(0, 0)}{h} = \lim_{h \to 0} \frac{\sqrt{h^2 u_1 u_2}}{h} = \lim_{h \to 0} \frac{|h|}{h} \sqrt{u_1 u_2} ,$$

where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. If $\mathbf{u} = \mathbf{i}$, $-\mathbf{i}$, $\mathbf{j}$ or $-\mathbf{j}$ then either $u_1 = 0$ or $u_2 = 0$ and this limit exists and equals 0. Conversely, if this limit exists then $u_1 u_2 \geq 0$, and

$$-\sqrt{u_1 u_2} = \lim_{h \to 0^-} \frac{|h|}{h} \sqrt{u_1 u_2} = \lim_{h \to 0^+} \frac{|h|}{h} \sqrt{u_1 u_2} = \sqrt{u_1 u_2} ,$$

so that in fact $u_1 u_2 = 0$, yielding $u_1 = 0$ or $u_2 = 0$. Therefore $\mathbf{u}$ must equal one of $\mathbf{i}$, $-\mathbf{i}$, $\mathbf{j}$ or $-\mathbf{j}$.

(b) If the one sided limit is used in the definition of $D_{\mathbf{u}}f(0, 0)$, i.e. taking only the limit as $h \to 0^+$, then the directional derivative is defined for directions with angle θ given in the interval $0 \leq \theta \leq \pi/2$ or $-\pi \leq \theta \leq -\pi/2$, i.e. in the first and third quadrants including the axes, and is given by $D_{\mathbf{u}}f(0, 0) = \sqrt{u_1 u_2}$.

5. Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin(xy)$ at $(1, 0)$ has value 1.

Solution

$\nabla f(x, y) = [\,2x + y\cos(xy)\,]\,\mathbf{i} + x\cos(xy)\,\mathbf{j}$, so $\nabla f(1, 0) = 2\,\mathbf{i} + \mathbf{j}$.
We want $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ such that $u_1^2 + u_2^2 = 1$ and

$$1 = \nabla f(1, 0) \cdot \mathbf{u} = 2u_1 + u_2.$$

Substituting $u_2 = 1 - 2u_1$ into $u_1^2 + u_2^2 = 1$ gives

$$1 = u_1^2 + (1 - 2u_1)^2 = 5u_1^2 - 4u_1 + 1 .$$

Hence $u_1(5u_1 - 4) = 0$, giving $u_1 = 0$ or $u_1 = 4/5$, and thus $u_2 = 1$ or $u_2 = -3/5$ respectively. The required directions are therefore those of the vectors $\mathbf{j}$ and $\frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$.

6. Find the greatest slope and the (two) directions one could begin to move to stay level if one is standing at the point

(a) $(3, 4, 13)$ on the surface $z = 1 + 2x\sqrt{y}$;

(b) $(2, 0, 0)$ on the surface $z = e^{-x}\sin y$.

Solution

(a) Let $f(x,y) = 1 + 2x\sqrt{y}$. We have $\nabla f(x,y) = 2\sqrt{y}\ \mathbf{i} + (x/\sqrt{y})\ \mathbf{j}$. Hence the greatest slope at $(3,4,13)$ is

$$|\nabla f(3,4)| = |4\,\mathbf{i} + \frac{3}{2}\mathbf{j}| = \frac{\sqrt{73}}{2},$$

and to stay level one moves in the direction perpendicular to the gradient of f at $(3,4)$, that is, in the direction of $\pm\left(\frac{3}{2}\mathbf{i} - 4\,\mathbf{j}\right)$.

(b) Let $g(x,y) = e^{-x}\sin y$. We have $\nabla g(x,y) = -e^{-x}\sin y\ \mathbf{i} + e^{-x}\cos y\ \mathbf{j}$, so $\nabla g(2,0) = e^{-2}\mathbf{j}$. The greatest slope is $|\nabla g(2,0)| = |e^{-2}\,\mathbf{j}| = e^{-2}$, and to stay level one moves in the direction of $\pm\mathbf{i}$.

7. Suppose you are climbing a hill whose shape is given by the equation

$$z = 1000 - 0.01x^2 - 0.02y^2,$$

where x, y, z are measured in metres, and you are standing at a point with coordinates $(50, 80, 847)$. The positive x axis points east and the positive y axis points north.

(a) If you walk due south, will you start to ascend or descend?

(b) If you walk northwest, will you start to ascend or descend?

(c) In which direction is the slope largest? What is the value of this slope? At what angle above the horizontal does the path in that direction begin?

(d) In which horizontal direction should you move to maintain a height of 847 metres?

Solution

Let $z = f(x,y) = 1000 - 0.01x^2 - 0.02y^2$. We have $\nabla f(x,y) = -0.02x\mathbf{i} - 0.04y\mathbf{j}$ and so $\nabla f(50,80) = -\mathbf{i} - 3.2\mathbf{j}$.

(a) In the direction of due south (that is, in the direction of $-\mathbf{j}$),

$$D_{-\mathbf{j}}f(50,80) = -\mathbf{j}\cdot(-\mathbf{i} - 3.2\mathbf{j}) = 3.2.$$

Since this is positive, you will start to ascend.

(b) In the north-west direction (that is, in the direction of the unit vector $\mathbf{u} = (-\mathbf{i}+\mathbf{j})/\sqrt{2}$),

$$D_{\mathbf{u}}f(50,80) = (-\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j})\cdot(-\mathbf{i} - 3.2\mathbf{j}) = 1/\sqrt{2} - 3.2/\sqrt{2} = -\frac{2.2}{\sqrt{2}}.$$

Since this is negative, you will start to descend.

(c) The slope is largest in the direction of $\nabla f(50,80) = -\mathbf{i} - 3.2\mathbf{j}$. The greatest slope is

$$|\nabla f(50,80)| = |-\mathbf{i} - 3.2\mathbf{j}| = \sqrt{1 + 3.2^2} \approx 3.35.$$

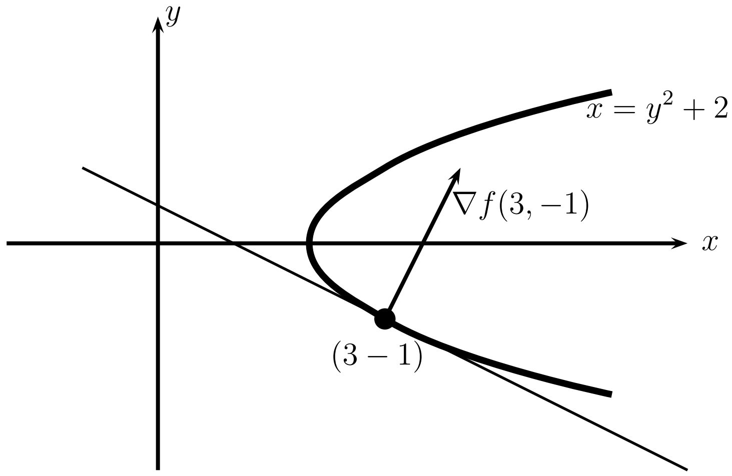
The corresponding angle above the horizontal path is approximately $\tan^{-1} 3.35$, or $73.4°$.

(d) To stay level, you should move perpendicular to $\nabla f(50,80)$, that is in the direction of $3.2\mathbf{i} - \mathbf{j}$ or $-3.2\mathbf{i} + \mathbf{j}$.

8. Let $f(x,y) = x - y^2$. Find $\nabla f(3,-1)$, and use it to find the parametric equation of the normal (perpendicular) line to the level curve $f(x,y) = 2$ at $(3,-1)$.

Solution

$\nabla f(x,y) = \mathbf{i} - 2y\,\mathbf{j}$, so $\nabla f(3,-1) = \mathbf{i} + 2\,\mathbf{j}$. The level curve $f(x,y) = 2$ is the parabola $x = y^2 + 2$. $\nabla f(3,-1)$ is perpendicular (normal) to the level curve $z = 2$ and passes through the point $(3,-1)$.
Thus parametric equations of the normal line are: $x = 3 + t$, $y = -1 + 2t$.



Extra Question

9. A function f of two variables is called *homogeneous of degree* $n \geq 1$ if

$$f(tx,ty) = t^n f(x,y)$$

for all t, x, y. Assume that all functions are well-behaved so that the chain rule applies.

(a) Verify that $g(x,y) = x^3 + xy^2 + y^3$ and $h(x,y) = (x^4 + y^4)^{3/2}$ are homogeneous of degrees 3 and 6 respectively.

(b) Suppose f is homogeneous of degree n and let $x = ta$, $y = tb$ where a and b are constants and t is a parameter. Put $F(t) = f(ta,tb)$. Differentiate $F(t)$ in two different ways (one using the chain rule) to conclude

$$nt^{n-1} f(a,b) = a\frac{\partial f}{\partial x}(ta,tb) + b\frac{\partial f}{\partial y}(ta,tb).$$

Set $t = 1$ and replace a by x and b by y to deduce *Euler's Theorem*:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y).$$

Solution

(a) We have

$$\begin{aligned} g(tx,ty) &= (tx)^3 + (tx)(ty)^2 + (ty)^3 \\ &= t^3(x^3 + xy^2 + y^3) \\ &= t^3 g(x,y), \end{aligned}$$

and

$$\begin{aligned} h(tx,ty) &= \left((tx)^4 + (ty)^4\right)^{3/2} \\ &= (t^4(x^4 + y^4))^{3/2} \\ &= t^6(x^4 + y^4)^{3/2} = t^6 h(x,y). \end{aligned}$$

(b) We have $F(t) = t^n f(a,b)$, so, on the one hand, $F'(t) = nt^{n-1} f(a,b)$, whilst on the other,

$$F'(t) = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y},$$

yielding

$$nt^{n-1}f(a,b) = a\frac{\partial f}{\partial x}(ta,tb) + b\frac{\partial f}{\partial y}(ta,tb).$$

In particular, taking $t = 1$, we get

$$nf(a,b) = a\frac{\partial f}{\partial x}(a,b) + b\frac{\partial f}{\partial y}(a,b)\ .$$

Finally using x and y as inputs we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y).$$

Solution to Question 1

First calculate $\nabla f(x,y) = (2x + 2e^{x+y})\mathbf{i} + 2e^{x+y}\mathbf{j}$. A unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$, and

$$D_{\mathbf{u}}f(x,y) = (\tfrac{1}{\sqrt{2}}\mathbf{i} - \tfrac{1}{\sqrt{2}}\mathbf{j}) \cdot ((2x + 2e^{x+y})\mathbf{i} + 2e^{x+y}\mathbf{j}) = \sqrt{2}x.$$

So the directional derivative at $(1,2)$ is $\sqrt{2}$.