

1. (This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.)

Express the following in the form  $x + iy$  (Cartesian or standard form):

- |                               |                             |
|-------------------------------|-----------------------------|
| (a) $(2 + 3i) + (4 - 5i)$ ;   | (b) $(1 + i)(1 - i)$ ;      |
| (c) $(2 + 3i) - (4 - 5i)$ ;   | (d) $\frac{1 + i}{1 - i}$ ; |
| (e) $\frac{1 + 2i}{3 - 4i}$ ; | (f) $(1 + i)^2$ ;           |
| (g) $i^9$ ;                   | (h) $i^{123} - 4i^8 - 4i$ . |

### Questions for the Tutorial

2. Solve the following equations in  $\mathbb{C}$ :

- |                              |                       |
|------------------------------|-----------------------|
| (a) $z^2 + 3z + 2 = 0$       | (b) $z^2 + z + 1 = 0$ |
| (c) $z^2 + 2\bar{z} + 1 = 0$ | (d) $z^4 = 16$        |

3. (a) Find all solutions of the equation  $z^2 + 3 + 4i = 0$  by setting  $z = a + bi$  for some real numbers  $a$  and  $b$ .

(b) Solve  $z^2 + z + 1 + i = 0$ . (*Hint*: Use your solution to the previous part.)

4. For all complex numbers  $z_1$  and  $z_2$ , prove that

- (a)  $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ , (b)  $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$ , (c)  $\overline{\overline{z_1}} = z_1$  if and only if  $z_1$  is real,  
and (d)  $\overline{\left(\frac{1}{z_1}\right)} = \frac{1}{\overline{z_1}}$  for  $z_1 \neq 0$ .

5. Sketch the following sets in the complex plane.

(*Hint*: Note that  $|z - c|$  is the distance between  $z$  and  $c$  in the complex plane, hence  $|z + c| = |z - (-c)|$  is the distance between  $z$  and  $-c$ .)

- |  |  |
|--|--|
| (a) $\{z \in \mathbb{C} \mid  z + i  = 5\}$ ,  | (b) $\{z \in \mathbb{C} \mid \frac{1}{2} \leq  z + i  < 1\}$ .               |
| (c) $\{z \in \mathbb{C} \mid  z  \leq 3\}$   | (d) $\{z \in \mathbb{C} \mid  z + i  > 2\}$                                  |
| (e) $\{z \in \mathbb{C} \mid \operatorname{Re} z < -1\}$   | (f) $\{z \in \mathbb{C} \mid \operatorname{Im} z \geq -1\}$                  |
| (g) $\{z \in \mathbb{C} \mid  z - i  \leq  z - 1 \}$   | (h) $\{z \in \mathbb{C} \mid \left  \frac{z - 1}{z - 2} \right  \leq 3\}$    |
| (i) $\{z \in \mathbb{C} \mid \operatorname{Im}(2z - \bar{z}(1 + i)) = 0 \text{ and } \operatorname{Re}(2z - \bar{z}(1 + i)) < 4\}$ | (j) $\{z \in \mathbb{C} \mid \operatorname{Im}(z^2) < \operatorname{Re} z\}$ |

6. (a) Write the following in *polar* form,  $z = r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta$ :

$$(i) \quad 1 + i \qquad (ii) \quad 1 + \sqrt{3}i$$

$$(iii) \quad 3\sqrt{3} + 3i$$

(b) Using your answers to the previous part, find the following, expressing your answers first in polar then Cartesian (standard) form.

$$(i) \quad (1 + i)^{11} \qquad (ii) \quad (1 + \sqrt{3}i)^7$$

$$(iii) \quad (3\sqrt{3} + 3i)^3 \qquad (iv) \quad \frac{1 + i}{1 + \sqrt{3}i}$$

$$(v) \quad \frac{3\sqrt{3} + 3i}{1 + i} \qquad (vi) \quad \frac{1 + \sqrt{3}i}{3\sqrt{3} + 3i}$$

7. Prove the *triangle inequality*  $|z_1 + z_2| \leq |z_1| + |z_2|$ , for all  $z_1, z_2 \in \mathbb{C}$ .

8. If the complex number  $z$  is imagined as a point in the complex plane, then its conjugate  $\bar{z}$  is the point obtained from  $z$  by reflecting in the real axis. What are the complex numbers obtained from  $z$  by the following geometric transformations?

- (a)  $180^\circ$  rotation about 0.                      (b) Reflection in the imaginary axis.  
(c)  $45^\circ$  clockwise rotation about 0.            (d) Reflection in the line  $y = x$ .

### Extra Questions

9. Express  $\cos 5\theta$  and  $\sin 5\theta$  in terms of  $\cos \theta$  and  $\sin \theta$ , respectively.

(Hint: Recall the binomial expansion:

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

Use this as well as de Moivre's Theorem to expand  $(\cos \theta + i \sin \theta)^5$ .)

10. (a) Let  $r$  be a real constant greater than 2. The set  $\{z \in \mathbb{C} \mid |z + 1| + |z - 1| = r\}$  is a curve in the plane. Describe it and then find its equation in terms of  $x$  and  $y$ .

(b) Next, assume that  $-2 < r < 2$  and  $r \neq 0$ .

Describe the curve  $\{z \in \mathbb{C} \mid |z + 1| - |z - 1| = r\}$  and find its equation.

### Solution to Question 1

1. (a)  $(2 + 3i) + (4 - 5i) = 6 - 2i$ ;    (b)  $(1 + i)(1 - i) = 1^2 - i^2 = 1 + 1 = 2$ ;

(c)  $(2 + 3i) - (4 - 5i) = -2 + 8i$ ;    (d)  $\frac{1 + i}{1 - i} = \frac{(1 + i)(1 + i)}{1^2 - i^2} = \frac{(1 + i)^2}{2} = i$ ;

(e)  $\frac{1 + 2i}{3 - 4i} = \frac{1}{5}(-1 + 2i)$ ;    (f)  $(1 + i)^2 = 2i$ ;

(g)  $i^9 = i^8 i = i$ ;    (h)  $i^{123} - 4i^8 - 4i = -4 - 5i$ .