

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Express the following complex numbers in Cartesian form:

(a) $2 \operatorname{cis} \frac{\pi}{4}$

(b) $-4 \operatorname{cis} \frac{\pi}{3}$

(c) $\operatorname{cis} \frac{\pi}{2} \operatorname{cis} \frac{\pi}{3} \operatorname{cis} \frac{\pi}{6}$

(d) $e^{-i\pi}$

(e) $e^{\ln 2 + i\pi}$

(f) $e^{1+i} e^{1-i} e^{-2-i\pi}$

Questions for the tutorial

2. Solve the following equations (leaving your answers in polar form) and plot the solutions in the complex plane:

(a) $z^5 = 1$

(b) $z^6 = -1$

(c) $z^3 + i = 0$

(d) $z^4 = 8\sqrt{2} + 8\sqrt{2}i$

(e) $z^5 + z^3 - z^2 - 1 = 0$, given that $z = i$ is a solution.

3. The complex sine and cosine functions are defined by the formulas

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C}.$$

- (a) Show that when z is real ($z = x$), $\sin z$ and $\cos z$ reduce to the familiar real sine and cosine functions.
- (b) Show that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$.
- (c) Is it true that $|\sin z| \leq 1$ and $|\cos z| \leq 1$, for all $z \in \mathbb{C}$? (*Hint:* You know these are true when z is real. See what happens when z is purely imaginary, $z = iy$.)

4. Find all solutions of the following equations:

(a) $e^z = i$

(b) $e^z = -10$

(c) $e^z = -1 - i\sqrt{3}$

(d) $e^{2z} = -i$

5. Sketch and describe the following sets and their images under the function $z \mapsto z^2$.

(a) The set of all points of the form $z = x + 2i$.

(b) The set of all points of the form $z = x + 2xi$.

(c) The set of all points on the upper half of the unit circle centred at the origin, that is, points z with polar coordinates (r, θ) such that $r = 1$ and $0 \leq \theta \leq \pi$.

(d) The set of all points on the unit circle centred at the origin, that is, points z with polar coordinates (r, θ) such that $r = 1$ and $-\pi < \theta \leq \pi$.

6. Sketch the following sets and their images under the function $z \mapsto e^z$:
- $\{z = x + iy \in \mathbb{C} \mid 0 < x < 2, y = \frac{\pi}{2}\}$;
 - $\{z \in \mathbb{C} \mid x = 1, |y| < \frac{\pi}{2}\}$;
 - $\{z \in \mathbb{C} \mid x < 0, \frac{\pi}{3} < y < \pi\}$;
 - $\{z = (1 + i)t \mid t \in \mathbb{R}\}$.
7. (a) Sketch the set $\{z \in \mathbb{C} \mid \frac{1}{2} < |z| < 4, 0 \leq \text{Arg}(z) \leq \frac{\pi}{4}\}$.
- (b) Sketch the image of the set in the w -plane under the function $z \mapsto w = \frac{1}{z}$.
- (c) An insect is crawling clockwise around the boundary of the set in the z -plane. Is its image crawling clockwise, or anticlockwise, in the w -plane? (If clockwise, we say the transformation is *orientation-preserving*; if anticlockwise, we say it is *orientation-reversing*.)
- (d) Now consider the function $z \mapsto w = \bar{z}$, the complex conjugate of z . Is it orientation-preserving or orientation-reversing?
8. Find all solutions of the equation $e^{2z} - (1 + 3i)e^z + i - 2 = 0$.

Extra Questions

9. This question demonstrates that complex numbers can be useful in solving cubic equations, even when all the solutions are real.
- Show that for any complex number w , there exists a nonzero complex number z such that $z + \frac{1}{z} = w$.
 - Use this substitution to solve the equation $w^3 - 3w - 1 = 0$.
10. Let n be a given positive integer. By a *primitive n th root of unity* we mean a solution η of $z^n = 1$ which has the property that its powers $\eta, \dots, \eta^{n-1}, \eta^n (= 1)$ are exactly the solutions of this equation in \mathbb{C} . For example, $e^{i\frac{2\pi}{n}}$ is a primitive n th root of unity.
- Find all primitive 6th roots of unity.
 - Find all primitive 5th roots of unity.
 - For which values of k , $0 \leq k \leq n - 1$, is $e^{i\frac{2\pi k}{n}}$ a primitive n th root of unity?

Solution to Question 1

1. (a) $2 \text{cis } \pi/4 = \sqrt{2} + \sqrt{2}i$ (b) $-4 \text{cis } \pi/3 = -2 - 2\sqrt{3}i$
- (c) $\text{cis } (\pi/2) \text{cis } (\pi/3) \text{cis } (\pi/6) = \text{cis } \pi = -1$ (d) $e^{-i\pi} = \text{cis } (-\pi) = -1$
- (e) $e^{\ln 2 + i\pi} = e^{\ln 2} \text{cis } \pi = -2$ (f) $e^{1+i}e^{1-i}e^{-2-i\pi} = e^{1+i+1-i-2-i\pi} = e^{-i\pi} = -1$