

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Express the following complex numbers in Cartesian form:

- |  |   |
|--|---|
| (a) $2 \operatorname{cis} \frac{\pi}{4}$   | (b) $-4 \operatorname{cis} \frac{\pi}{3}$ |
| (c) $\operatorname{cis} \frac{\pi}{2} \operatorname{cis} \frac{\pi}{3} \operatorname{cis} \frac{\pi}{6}$ | (d) $e^{-i\pi}$                           |
| (e) $e^{\ln 2 + i\pi}$   | (f) $e^{1+i} e^{1-i} e^{-2-i\pi}$         |

### Questions for the tutorial

2. Solve the following equations (leaving your answers in polar form) and plot the solutions in the complex plane:

- (a)  $z^5 = 1$
- (b)  $z^6 = -1$
- (c)  $z^3 + i = 0$
- (d)  $z^4 = 8\sqrt{2} + 8\sqrt{2}i$
- (e)  $z^5 + z^3 - z^2 - 1 = 0$ , given that  $z = i$  is a solution.

3. The complex sine and cosine functions are defined by the formulas

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad z \in \mathbb{C}.$$

- (a) Show that when  $z$  is real ( $z = x$ ),  $\sin z$  and  $\cos z$  reduce to the familiar real sine and cosine functions.
- (b) Show that  $\sin^2 z + \cos^2 z = 1$  for all  $z \in \mathbb{C}$ .
- (c) Is it true that  $|\sin z| \leq 1$  and  $|\cos z| \leq 1$ , for all  $z \in \mathbb{C}$ ? (*Hint:* You know these are true when  $z$  is real. See what happens when  $z$  is purely imaginary,  $z = iy$ .)

4. Find all solutions of the following equations:

- |                            |                   |
|----------------------------|-------------------|
| (a) $e^z = i$              | (b) $e^z = -10$   |
| (c) $e^z = -1 - i\sqrt{3}$ | (d) $e^{2z} = -i$ |

5. Sketch and describe the following sets and their images under the function  $z \mapsto z^2$ .

- (a) The set of all points of the form  $z = x + 2i$ .
- (b) The set of all points of the form  $z = x + 2xi$ .
- (c) The set of all points on the upper half of the unit circle centred at the origin, that is, points  $z$  with polar coordinates  $(r, \theta)$  such that  $r = 1$  and  $0 \leq \theta \leq \pi$ .
- (d) The set of all points on the unit circle centred at the origin, that is, points  $z$  with polar coordinates  $(r, \theta)$  such that  $r = 1$  and  $-\pi < \theta \leq \pi$ .

6. Sketch the following sets and their images under the function  $z \mapsto e^z$ :

(a)  $\{z = x + iy \in \mathbb{C} \mid 0 < x < 2, y = \frac{\pi}{2}\}$ ;

(b)  $\{z \in \mathbb{C} \mid x = 1, |y| < \frac{\pi}{2}\}$ ;

(c)  $\{z \in \mathbb{C} \mid x < 0, \frac{\pi}{3} < y < \pi\}$ ;

(d)  $\{z = (1 + i)t \mid t \in \mathbb{R}\}$ .

7. (a) Sketch the set  $\{z \in \mathbb{C} \mid \frac{1}{2} < |z| < 4, 0 \leq \text{Arg}(z) \leq \frac{\pi}{4}\}$ .

(b) Sketch the image of the set in the  $w$ -plane under the function  $z \mapsto w = \frac{1}{z}$ .

(c) An insect is crawling clockwise around the boundary of the set in the  $z$ -plane. Is its image crawling clockwise, or anticlockwise, in the  $w$ -plane? (If clockwise, we say the transformation is *orientation-preserving*; if anticlockwise, we say it is *orientation-reversing*.)

(d) Now consider the function  $z \mapsto w = \bar{z}$ , the complex conjugate of  $z$ . Is it orientation-preserving or orientation-reversing?

8. Find all solutions of the equation  $e^{2z} - (1 + 3i)e^z + i - 2 = 0$ .

### Extra Questions

9. This question demonstrates that complex numbers can be useful in solving cubic equations, even when all the solutions are real.

(a) Show that for any complex number  $w$ , there exists a nonzero complex number  $z$  such that  $z + \frac{1}{z} = w$ .

(b) Use this substitution to solve the equation  $w^3 - 3w - 1 = 0$ .

10. Let  $n$  be a given positive integer. By a *primitive  $n$ th root of unity* we mean a solution  $\eta$  of  $z^n = 1$  which has the property that its powers  $\eta, \dots, \eta^{n-1}, \eta^n (= 1)$  are exactly the solutions of this equation in  $\mathbb{C}$ . For example,  $e^{i\frac{2\pi}{n}}$  is a primitive  $n$ th root of unity.

(a) Find all primitive 6th roots of unity.

(b) Find all primitive 5th roots of unity.

(c) For which values of  $k$ ,  $0 \leq k \leq n - 1$ , is  $e^{i\frac{2\pi k}{n}}$  a primitive  $n$ th root of unity?

### Solution to Question 1

1. (a)  $2 \text{cis } \pi/4 = \sqrt{2} + \sqrt{2}i$  (b)  $-4 \text{cis } \pi/3 = -2 - 2\sqrt{3}i$

(c)  $\text{cis } (\pi/2) \text{cis } (\pi/3) \text{cis } (\pi/6) = \text{cis } \pi = -1$  (d)  $e^{-i\pi} = \text{cis } (-\pi) = -1$

(e)  $e^{\ln 2 + i\pi} = e^{\ln 2} \text{cis } \pi = -2$  (f)  $e^{1+i}e^{1-i}e^{-2-i\pi} = e^{1+i+1-i-2-i\pi} = e^{-i\pi} = -1$