

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Sketch the graph of the function:

$$f(x) = \begin{cases} 0 & x < 0, \\ 1 & x = 0, \\ x + 2 & x > 0. \end{cases}$$

Find  $\lim_{x \rightarrow 0^-} f(x)$  and  $\lim_{x \rightarrow 0^+} f(x)$  (no need for formal proofs). Does  $\lim_{x \rightarrow 0} f(x)$  exist?

### Questions for the tutorial

2. Sketch the function with formula

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 2x + 1 & \text{if } x > 0. \end{cases}$$

Find suitable values of  $\delta$  such that whenever  $0 < |x| < \delta$ , we have

- (a)  $|f(x) - 1| < 0.01$ , (b)  $|f(x) - 1| < 0.001$ , (c)  $|f(x) - 1| < \epsilon$ , where  $\epsilon > 0$ .

3. Find the following limits using one or more of the limit laws.

(a)  $\lim_{x \rightarrow 3} \frac{x^2 + 3x + 2}{4x^2 - x + 1}$

(b)  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$

(c)  $\lim_{x \rightarrow 1} \frac{x^2 - 3x + 2}{x^3 - 1}$

(d)  $\lim_{x \rightarrow 0} x^2 \cos \frac{2}{x}$

(e)  $\lim_{x \rightarrow 0} \frac{\sqrt{3 + 2x} - \sqrt{3}}{x}$

(f)  $\lim_{x \rightarrow \infty} \frac{x + \sin^3 x}{2x - 1}$

(g)  $\lim_{x \rightarrow \infty} \sqrt{\frac{3 - x}{4 - x}}$

(h)  $\lim_{x \rightarrow \infty} \sqrt{\frac{3 - x}{4 - x^2}}$

(i)  $\lim_{x \rightarrow \infty} (\sqrt{x} - \sqrt{x + 1})$

4. Prove the following results using the  $\epsilon, \delta$  definition:

(a)  $\lim_{x \rightarrow a} c = c$

(b)  $\lim_{x \rightarrow 4} f(x) = -3$ , where  $f(x) = \begin{cases} 5 - 2x & \text{if } x \neq 4, \\ 100 & \text{if } x = 4. \end{cases}$

(c)  $\lim_{x \rightarrow 0} g(x) = 0$ , where  $g(x) = \begin{cases} 3x & \text{if } x \text{ is rational,} \\ 7x & \text{if } x \text{ is irrational.} \end{cases}$

5. The function  $f$  is defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

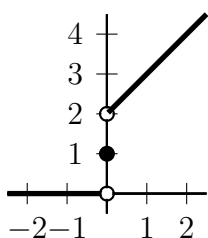
Using the Squeeze Law, prove that  $\lim_{x \rightarrow 0} f(x) = 0$ .

6. (a) Give an example of a function  $f$  for which  $\lim_{x \rightarrow 0} f(x)^2$  exists but  $\lim_{x \rightarrow 0} f(x)$  does not.  
 (b) Give an example of a function  $f$  for which  $\lim_{x \rightarrow 0} f(x^2)$  exists but  $\lim_{x \rightarrow 0} f(x)$  does not.
7. Suppose  $f$  has domain  $\mathbb{R}$ . To say that  $\lim_{n \rightarrow \infty} f(n) = \ell$  (where  $n$  takes only integer values) means that for any  $\epsilon > 0$ , there exists  $M$  such that whenever  $n$  is an integer and  $n > M$ , then  $|f(n) - \ell| < \epsilon$ . Give an example of a function  $f$  with domain  $\mathbb{R}$  such that  $\lim_{n \rightarrow \infty} f(n)$  exists in this sense, but  $\lim_{x \rightarrow \infty} f(x)$  (where  $x$  takes real values) does not exist.
8. Using the  $\epsilon, \delta$  definition of limit, prove that if the limit of a function exists as  $x \rightarrow a$ , then the limit is unique. To be precise, prove that if  $\lim_{x \rightarrow a} f(x) = \ell$  and  $\lim_{x \rightarrow a} f(x) = m$ , then  $\ell = m$ . (*Hint*: Assume that  $\ell \neq m$  and obtain a contradiction by setting  $\epsilon = \frac{|\ell - m|}{2}$ .)

### Extra Questions

9. Students often have difficulty remembering the  $\epsilon, \delta$  definition of the statement  $\lim_{x \rightarrow a} f(x) = \ell$ . For each of the following misremembered versions, work out what it means and why it is not the right definition.
- (a) For each  $\epsilon > 0$ , there exists  $\delta > 0$  such that whenever  $0 < |x - a| < \delta$ , we have  $0 < |f(x) - \ell| < \epsilon$ .
- (b) For each  $\epsilon > 0$ , there exists  $\delta > 0$  such that whenever  $|x - a| < \delta$ , we have  $|f(x) - \ell| < \epsilon$ .
- (c) For each  $\epsilon$ , there exists  $\delta > 0$  such that whenever  $0 < |x - a| < \delta$ , we have  $|f(x) - \ell| < \epsilon$ .
- (d) For each  $\epsilon > 0$ , there exists  $\delta$  such that whenever  $0 < |x - a| < \delta$ , we have  $|f(x) - \ell| < \epsilon$ .
- (e) For each  $\epsilon > 0$ , there exists  $\delta > 0$  such that whenever  $0 < |x - a| < \delta$ , we have  $|f(x) - \ell| > \epsilon$ .
- (f) For each  $\epsilon > 0$ , there exists  $\delta > 0$  such that whenever  $0 < |x - a| < \epsilon$ , we have  $|f(x) - \ell| < \delta$ .
- (g) For each  $\delta > 0$ , there exists  $\epsilon > 0$  such that whenever  $0 < |x - a| < \delta$ , we have  $|f(x) - \ell| < \epsilon$ .
- (h) For each  $\delta > 0$ , there exists  $\epsilon > 0$  such that whenever  $0 < |x - a| < \epsilon$ , we have  $|f(x) - \ell| < \delta$ .
10. Prove that  $\lim_{x \rightarrow 0} f(x)$  does not exist, where  $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$

### Solution to Question 1



We have  $\lim_{x \rightarrow 0^-} f(x) = 0$  and  $\lim_{x \rightarrow 0^+} f(x) = 2$ .

Since the limits from the left and the right are not equal,  $\lim_{x \rightarrow 0} f(x)$  does not exist.