THE UNIVERSITY OF SYDNEY

MATH1901 DIFFERENTIAL CALCULUS (ADVANCED)

Semester 1	**Tutorial Week 8**	2009

(*These preparatory questions should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

1. The function f is defined by $f(x) = |x-1|$. Sketch its graph. Observe that there is no value c such that $f(3) - f(0) = f'(c)(3-0)$. Why does this not contradict the Mean Value Theorem?

2. Using l'Hôpital's rule, find $\displaystyle\lim_{x\to\frac{3\pi}{2}} \frac{\cos x}{x-(3\pi/2)}$.

Questions for the tutorial

3. Use the Mean Value Theorem to prove the following inequalities:

(a) $|\cos y - \cos x| \le |y - x|$, for all real numbers x and y;

(b) $|\sinh x| \ge |x|$ for all real x;

(c) $e^x \ge 1 + x$ for all real x.

4. Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x. Use the Mean Value Theorem to show that the largest possible value of $f(2)$ is 7.

5. The road between two towns, A and B, is 110 km long. You left A to drive to B at the same time as I left B to drive to A. We met exactly 30 minutes later. Use the Mean Value Theorem to prove that at least one of us exceeded the speed limit, 100 km/h, by at least 10 km/h.

6. Find the following limits.

(a) $\displaystyle\lim_{x\to-1} \frac{x^6-1}{x^4-1}$ (b) $\displaystyle\lim_{x\to\pi} \frac{\tan x}{x}$ (c) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\ln(\ln x)}$

(d) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^{1/100}}$ (e) $\displaystyle\lim_{x\to 0} \frac{e^x-1-x}{x^2}$ (f) $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}}$

7. Use induction on n and l'Hôpital's rule to prove that $\displaystyle\lim_{x\to 0^+} x(\ln x)^n = 0$ for $n = 0, 1, 2, \cdots$.

8. Find the following limits.

(a) $\displaystyle\lim_{x\to\infty} x^{1/x}$ (*Hint*: Set $y = x^{1/x}$, and compute $\displaystyle\lim_{x\to\infty} \ln y$.)

(b) $\displaystyle\lim_{x\to 0^+} x^{1/x}$

(c) $\displaystyle\lim_{x\to\infty} (1+e^{-x})^x$

(d) $\lim_{x\to\frac{\pi}{2}^-} (\tan x)^{\cos x}$

(e) $\lim_{x\to 0^+} (\sinh \frac{4}{x})^x$

Extra Questions

9. What is wrong with the following "proof" of the Cauchy Mean Value Theorem?

CMVT: If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there is a number x in (a, b) such that $(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x)$.

"Proof": Applying the Mean Value Theorem to f and g separately, we find that there is an x in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(x) \quad \text{and} \quad \frac{g(b) - g(a)}{b - a} = g'(x).$$

Therefore

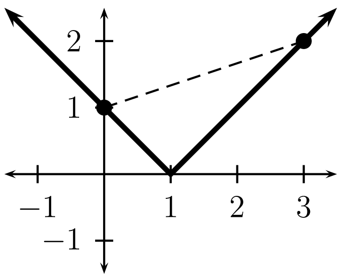
$$(f(b) - f(a))g'(x) = (b - a)f'(x)g'(x) = (g(b) - g(a))f'(x),$$

which proves the Theorem.

10. Consider the statement: "if f and g are differentiable, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} \frac{f(x)}{g(x)} = L$, then $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$". Show that this is false by giving a counter-example.

Solution to Question 1

The graph of f is as follows.



Now $f(3) - f(0) = 1$, and there is no value c such that $1 = 3f'(c)$. This does not contradict the Mean Value Theorem because f is not differentiable on the interval $(0, 3)$. (Specifically, $f'(1)$ does not exist.)

Solution to Question 2

Let $f(x) = \cos x$ and $g(x) = x - \frac{3\pi}{2}$. Then $\lim_{x\to\frac{3\pi}{2}} f(x) = \lim_{x\to\frac{3\pi}{2}} g(x) = 0$, and

$$\lim_{x\to\frac{3\pi}{2}} \frac{f'(x)}{g'(x)} = \lim_{x\to\frac{3\pi}{2}} (-\sin x) = -\sin\frac{3\pi}{2} = 1.$$

So l'Hôpital's rule says that $\lim_{x\to\frac{3\pi}{2}} \frac{\cos x}{x - (3\pi/2)} = 1$.