

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Given the Taylor formula $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + R_n(x)$, where $R_n(x) = \frac{(-1)^n x^{n+1}}{(n+1)(1+c)^{n+1}}$ for some c between 0 and x ,

- (a) find the Taylor polynomial of order $n+2$ for $x^2 \ln(1+x)$ about the point 0,
(b) find the Taylor polynomial of order n for $\ln(1-x)$ about the point 0.

Questions for the tutorial

2. Find the Taylor polynomial $T_5(x)$ of order five about $x=0$ for each of the following functions. Write down the remainder term $R_5(x)$ in each case, and estimate the size of the error if $T_5(1)$ is used as an approximation to $f(1)$.

(a) $f(x) = \sqrt{1+x}$ (b) $f(x) = \cosh x$

3. (a) Find the Taylor polynomial of order 4 about $x=0$ for $\frac{1}{1+x}$.
(b) Find the Taylor polynomial of order 5 about $x=0$ for $\ln(1+x)$.
(c) What relationship can you see between the two polynomials above? Why might you expect such a relationship?

4. (a) Find the Taylor polynomials of orders 2 and 4 about $x = \frac{\pi}{2}$, for $f(x) = \cos x$. Use these polynomials to estimate $\cos \frac{4\pi}{7}$ and $\cos \frac{5\pi}{7}$. Compare your results with those obtained from a calculator.
(b) Use Taylor polynomials of order 3 about $x = \frac{\pi}{2}$ and $x = \pi$ to estimate $\sin 3$. Which is the better approximation?

5. Find the Taylor polynomial of order 2 for $f(x) = \tan^{-1} x$ about 0, and write down the remainder term. Using this information, show that $\int_0^{0.1} \tan^{-1} x \, dx$ lies between 0.00499 and 0.00501.

6. You are given that the Taylor polynomial $T_3(x)$ of order 3 for $\sqrt{1+x}$, about 0, is $T_3(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$, with $R_3(x) = -\frac{15}{16}(1+c)^{-\frac{7}{2}} \frac{x^4}{4!}$, for some c between 0 and x .

- (a) Write down the Taylor polynomial of order 9 about 0 for $\sqrt{1+x^3}$.
(b) Use your answer to the previous part to find an approximation to the integral $\int_0^1 \sqrt{1+x^3} \, dx$. Find an upper bound for the error involved.

7. Use the Taylor polynomial of order 3 for $\sinh x$ about 0 to estimate $\int_0^1 \sinh x \, dx$. Determine the accuracy of your estimate and compare it to the value of the integral found using your calculator (the integral equals $\cosh 1 - \cosh 0 = \frac{e + e^{-1}}{2} - 1$). What difference would it make to the accuracy if we had used the Taylor polynomial of order 4?

Extra Questions

8. (a) The hyperbolic tan function is defined by $\tanh x = \frac{\sinh x}{\cosh x}$. It is a bijection from \mathbb{R} to $(-1, 1)$. Find a formula for $\tanh^{-1} x$ in terms of natural logarithms and use it to show that $\ln 2 = 2 \tanh^{-1} \frac{1}{3}$.
- (b) Find the Taylor polynomial of order $2n$ for $\tanh^{-1} x$ about the point 0 and write down its remainder term. (*Hint*: use the Taylor formulas for $\ln(1 \pm x)$ given in Question 1.)
- (c) Use the $n = 8$ case of the previous part to estimate $\ln 2$. Show that the error is less than 5×10^{-7} .
9. Consider the function given by

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Show that f is differentiable and that $f'(0) = 0$. Then show that f' is differentiable and that $f''(0) = 0$. In fact, it turns out that f is differentiable any number of times and its derivative at zero is always zero! This means that its Taylor polynomial about 0 of order n , for any n , is the zero polynomial. This function is “all remainder”.

Solution to Question 1

- (a) We multiply the Taylor formula for $\ln(1+x)$ by x^2 to obtain

$$x^2 \ln(1+x) = x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \dots + (-1)^{n-1} \frac{x^{n+2}}{n} + \frac{(-1)^n x^{n+3}}{(n+1)(1+c)^{n+1}}.$$

This equation shows that the polynomial $T(x) = x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \dots + (-1)^{n-1} \frac{x^{n+2}}{n}$ of degree $n+2$ has the property that

$$\lim_{x \rightarrow 0} \frac{x^2 \ln(1+x) - T(x)}{x^{n+2}} = 0,$$

so it must be the Taylor polynomial of order $n+2$ about 0, for $x^2 \ln(1+x)$.

- (b) We replace x by $-x$ in the formula for $\ln(1+x)$:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^n}{n} - \frac{x^{n+1}}{(n+1)(1+c)^{n+1}},$$

for some c between 0 and $-x$. By similar reasoning to part (a), $-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^n}{n!}$ must be the Taylor polynomial of order n about 0, for $\ln(1-x)$.