

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Compute the partial derivatives  $f_x(x, y)$ ,  $f_y(x, y)$  of the following functions  $f(x, y)$ .

(a)  $xy^3$                                       (b)  $\sin(2x + 3y)$                                       (c)  $\ln(x + \sqrt{x^2 + y^2})$

**Questions for the tutorial**

2. Find the limit, if it exists, or show that the limit does not exist.

(a)  $\lim_{(x,y) \rightarrow (2,3)} (x^2y^2 - 2xy^5 + 3y)$                                       (b)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y^3 + x^3y^2 - 5}{2 - xy}$

(c)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x - y}{x^2 + y^2}$                                       (d)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + xy^2}{x^2 + y^2}$

3. Consider the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}, \text{ defined for } (x, y) \neq (0, 0).$$

Is it possible to define  $f(0, 0)$  so that  $f$  is continuous at  $(0, 0)$ ?

4. Decide whether the limits exist.

(a)  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4}$                                       (b)  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} \sin \frac{1}{x^2 + y^4}$

(c)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$                                       (d)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$

5. Define  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  as follows:

$$f(x, y) = \begin{cases} 1 & \text{if } x = y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Show that  $f$  is not continuous at  $(0, 0)$  but both  $f_x$  and  $f_y$  exist at  $(0, 0)$ .

6. Verify that the functions given by the following formulas are solutions of the Laplace equation  $f_{xx} + f_{yy} = 0$ .

(a)  $x^2 - y^2$                                       (b)  $2xy$                                       (c)  $e^x \cos y$                                       (d)  $e^x \sin y$

7. Suppose that  $f$  is a differentiable function of one variable. Show that if  $z = f\left(\frac{x}{y}\right)$ , then

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.$$

8. Find the equation of the tangent plane to the surface  $z = e^x \ln y$  at  $(3, 1, 0)$ .
9. Find the single point at which the tangent plane to the surface  $z = x^2 + 2xy + 2y^2 - 6x + 8y$  is horizontal.

### Extra Question

10. Use the  $\epsilon, \delta$  definition of the limit of a function of two variables to show that

$$\lim_{(x,y) \rightarrow (1,2)} x^2 + y = 3.$$

### Solution to Question 1

(a)  $f_x = y^3, f_y = 3xy^2$

(b)  $f_x = 2 \cos(2x + 3y), f_y = 3 \cos(2x + 3y)$

(c)  $f_x = \frac{1 + x(x^2 + y^2)^{-1/2}}{x + \sqrt{x^2 + y^2}} = \frac{1}{\sqrt{x^2 + y^2}}, f_y = \frac{y}{(x + \sqrt{x^2 + y^2})\sqrt{x^2 + y^2}}$