

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Compute the partial derivatives $f_x(x, y)$, $f_y(x, y)$ of the following functions $f(x, y)$.

(a) xy^3 (b) $\sin(2x + 3y)$ (c) $\ln(x + \sqrt{x^2 + y^2})$

Questions for the tutorial

2. Find the limit, if it exists, or show that the limit does not exist.

(a) $\lim_{(x,y) \rightarrow (2,3)} (x^2y^2 - 2xy^5 + 3y)$ (b) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y^3 + x^3y^2 - 5}{2 - xy}$

(c) $\lim_{(x,y) \rightarrow (0,0)} \frac{x - y}{x^2 + y^2}$ (d) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + xy^2}{x^2 + y^2}$

3. Consider the function

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}, \text{ defined for } (x, y) \neq (0, 0).$$

Is it possible to define $f(0, 0)$ so that f is continuous at $(0, 0)$?

4. Decide whether the limits exist.

(a) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4}$ (b) $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} \sin \frac{1}{x^2 + y^4}$

(c) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ (d) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$

5. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ as follows:

$$f(x, y) = \begin{cases} 1 & \text{if } x = y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is not continuous at $(0, 0)$ but both f_x and f_y exist at $(0, 0)$.

6. Verify that the functions given by the following formulas are solutions of the Laplace equation $f_{xx} + f_{yy} = 0$.

(a) $x^2 - y^2$ (b) $2xy$ (c) $e^x \cos y$ (d) $e^x \sin y$

7. Suppose that f is a differentiable function of one variable. Show that if $z = f\left(\frac{x}{y}\right)$, then

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.$$

8. Find the equation of the tangent plane to the surface $z = e^x \ln y$ at $(3, 1, 0)$.
9. Find the single point at which the tangent plane to the surface $z = x^2 + 2xy + 2y^2 - 6x + 8y$ is horizontal.

Extra Question

10. Use the ϵ, δ definition of the limit of a function of two variables to show that

$$\lim_{(x,y) \rightarrow (1,2)} x^2 + y = 3.$$

Solution to Question 1

(a) $f_x = y^3, f_y = 3xy^2$

(b) $f_x = 2 \cos(2x + 3y), f_y = 3 \cos(2x + 3y)$

(c) $f_x = \frac{1 + x(x^2 + y^2)^{-1/2}}{x + \sqrt{x^2 + y^2}} = \frac{1}{\sqrt{x^2 + y^2}}, f_y = \frac{y}{(x + \sqrt{x^2 + y^2})\sqrt{x^2 + y^2}}$