

1. (*This question is a preparatory question and should be attempted before the tutorial. Answers are provided at the end of the sheet – please check your work.*)

Find the directional derivative of $f(x, y) = x^2 + 2e^{x+y}$ in the direction of $\mathbf{v} = \mathbf{i} - \mathbf{j}$ at the point $(1, 2)$.

Questions for the tutorial

2. Use the formula $\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}$ to find an expression for $\frac{dy}{dx}$ where y is defined implicitly as a function of x by the equation $x^3 + y^3 = 3xy$. Hence evaluate the slope of the tangent to the curve $x^3 + y^3 = 3xy$ at the point $(2/3, 4/3)$.

3. Let $f(x, y) = 1 + 2x\sqrt{y}$ and $g(x, y) = e^{-x} \sin y$.

(a) Find $\nabla f(x, y)$, $\nabla f(3, 4)$, $\nabla g(x, y)$, $\nabla g(2, 0)$.

(b) Let $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$. Determine the unit vector $\hat{\mathbf{v}}$. Hence find $D_{\hat{\mathbf{v}}}f(x, y)$ and also the special case $D_{\hat{\mathbf{v}}}f(3, 4)$. Similarly, if $\mathbf{w} = 3\mathbf{i} + 2\mathbf{j}$, find $D_{\hat{\mathbf{w}}}g(x, y)$ and $D_{\hat{\mathbf{w}}}g(2, 0)$.

4. Instead of the one-sided limit used in the definition of the directional derivative in this course, many texts use the following two-sided limit:

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector and h may be either positive or negative.

(a) Let $f(x, y) = \sqrt{xy}$ and let \mathbf{u} be a unit vector. Prove that $D_{\mathbf{u}}f(0, 0)$, defined using the two-sided limit above, exists if and only if $\mathbf{u} = \mathbf{i}$, $-\mathbf{i}$, \mathbf{j} or $-\mathbf{j}$.

(b) Now use our one-sided definition for the limit and find all directions for which $D_{\mathbf{u}}f(0, 0)$ exists.

5. Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin(xy)$ at $(1, 0)$ has value 1.

6. Find the greatest slope and the (two) directions one could begin to move to stay level if one is standing at the point

(a) $(3, 4, 13)$ on the surface $z = 1 + 2x\sqrt{y}$;

(b) $(2, 0, 0)$ on the surface $z = e^{-x} \sin y$.

7. Suppose you are climbing a hill whose shape is given by the equation

$$z = 1000 - 0.01x^2 - 0.02y^2,$$

where x, y, z are measured in metres, and you are standing at a point with coordinates $(50, 80, 847)$. The positive x axis points east and the positive y axis points north.

(a) If you walk due south, will you start to ascend or descend?

- (b) If you walk northwest, will you start to ascend or descend?
- (c) In which direction is the slope largest? What is the value of this slope? At what angle above the horizontal does the path in that direction begin?
- (d) In which horizontal direction should you move to maintain a height of 847 metres?
8. Let $f(x, y) = x - y^2$. Find $\nabla f(3, -1)$, and use it to find the parametric equation of the normal (perpendicular) line to the level curve $f(x, y) = 2$ at $(3, -1)$.

Extra Question

9. A function f of two variables is called *homogeneous of degree* $n \geq 1$ if

$$f(tx, ty) = t^n f(x, y)$$

for all t, x, y . Assume that all functions are well-behaved so that the chain rule applies.

- (a) Verify that $g(x, y) = x^3 + xy^2 + y^3$ and $h(x, y) = (x^4 + y^4)^{3/2}$ are homogeneous of degrees 3 and 6 respectively.
- (b) Suppose f is homogeneous of degree n and let $x = ta$, $y = tb$ where a and b are constants and t is a parameter. Put $F(t) = f(ta, tb)$. Differentiate $F(t)$ in two different ways (one using the chain rule) to conclude

$$nt^{n-1}f(a, b) = a\frac{\partial f}{\partial x}(ta, tb) + b\frac{\partial f}{\partial y}(ta, tb).$$

Set $t = 1$ and replace a by x and b by y to deduce *Euler's Theorem*:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

Solution to Question 1

First calculate $\nabla f(x, y) = (2x + 2e^{x+y})\mathbf{i} + 2e^{x+y}\mathbf{j}$. A unit vector in the direction of \mathbf{v} is $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$, and

$$D_{\mathbf{u}}f(x, y) = \left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\right) \cdot ((2x + 2e^{x+y})\mathbf{i} + 2e^{x+y}\mathbf{j}) = \sqrt{2}x.$$

So the directional derivative at $(1, 2)$ is $\sqrt{2}$.