The University of Sydney

MATH1901/06 Differential Calculus (Advanced)

Semester 1 **Practice Questions for Quiz 2** 2009

Quiz 2 will be held in the tutorials in **Week 11** (Monday 18 May 2009).

The quiz questions will be based on material covered in the lectures during Weeks 6–9, which corresponds to material covered in tutorials in Weeks 7–10.

Topics to be tested include derivatives, critical points, extrema, corner points, vertical tangents, l'Hôpital's rule, Rolle's theorem, mean value theorem, Taylor's formula with remainder, applications of Taylor polynomials (approximations, l'Hôpital-type limits), new Taylor polynomials from old, and level curves for functions of two variables.

The quiz will run for 30 minutes. You may use a non-programmable calculator. No other materials are permitted.

Solutions to these problems appear below after Question 10. (Note that there have been a few minor changes to the questions since they were first posted.)

The actual quiz questions will be considerably shorter than these practice questions and will not have multiple parts. Some quiz questions will be multiple choice.

1. (a) Calculate the following limits. (Some may be assisted by l'Hôpital's rule, some by Taylor polynomials, some by neither of these methods.)

$$\lim_{x\to 1}\frac{x^3-5x+4}{x^3-4x+3}, \qquad \lim_{x\to 0^+}(\sinh x)^{1/x}, \qquad \lim_{x\to\infty}(\sinh x)^{1/x},$$

$$\lim_{x\to 2}\frac{x^x-4}{2^x-4}, \qquad \lim_{x\to 0^+}x^x, \qquad \lim_{x\to 0^+}\frac{d}{dx}\,x^x,$$

$$\lim_{x\to 0}\frac{(ax-\sin ax)^2}{(1-\cos bx)^3}, \qquad \lim_{x\to 0}\frac{e^{-1/x^2}}{x^n}, \quad n=0,1,2,3,\ldots.$$

(b) Show that the following sequences are increasingly rapidly growing:

$$\{n^{100}\},\quad \{n^{\ln n}\},\quad \{e^{\sqrt{n}}\},\quad \{n^{\sqrt{n}}\},\quad \{2^n\},\quad \{n!\},\quad \{n^n\},\quad \{2^{n^2}\}.$$

2. Show from first principles that $(d/dx)\sin x=\cos x$ and $(d/dx)\cos x=-\sin x$. You will need to use the limit $(\sin\theta)/\theta\to 1$ as $\theta\to 0$, proved in lectures. (An appendix to the solutions will contain a first-principles derivation of $(d/dx)\log_b x$.)

3. Suppose $f(x)$ is a function that is continuous on $[0,1]$ and differentiable on $(0,1)$. Which of the following statements are TRUE and which are FALSE?

(a) $f(x)$ must have a right derivative at $x = 0$.

(b) If $f'(x)$ tends to a finite limit L as $x \to 0^+$, then $f(x)$ necessarily has a right derivative at $x = 0$ whose value is L.

(c) If $f'(x)$ tends to either $+\infty$ or $-\infty$ as $x \to 0^+$, then the graph of $y = f(x)$ on $[0,1]$ must have a one-sided vertical tangent at $x = 0$.

(d) If $f'(x)$ does not tend to a finite limit as $x \to 0^+$, then $f(x)$ cannot have a right derivative at $x = 0$.

(e) If $f(x)$ satisfies the inequality $ax \leq f(x) \leq bx$ on $[0,1]$ for particular real numbers a, b, $a < b$, then $f'(x)$ satisfies $a \leq f'(x) \leq b$ on $(0,1)$.

(f) There exists a point $c \in (0,1)$ such that $f'(c) = f(1) - f(0)$.

(g) The point c in the previous part is unique when it exists.

4. (a) Show that the graph of $y = x^{1/3}$ has a vertical tangent and an inflection at $x = 0$.

(b) Suppose the function $f(x)$ has a zero first derivative and a positive second derivative at $x = x_0$, but is possibly not twice differentiable anywhere else. Prove that $f(x)$ has a local minimum at $x = x_0$.

(c) Show that the graph of $y = \sin^{-1}(1 - x^2)$ has a corner point at $x = 0$.

(d) Find all values of the positive real parameters a and b such that the function,

$$f(x) = \begin{cases} |x|^a \sin(1/|x|^b), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

has a second derivative $f''(0)$ at $x = 0$. When is $f''(x)$ continuous at $x = 0$?

(e) Show that $(1 + 1/x)^x$ is increasing for all $x > 0$.

(f) One of the versions of l'Hôpital's rule states that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

whenever the limit on the right-hand side exists and $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$. Consider the case,

$$f(x) = x + \sin x \cos x, \qquad g(x) = f(x)e^{\sin x}.$$

Here $f(x)/g(x)$ oscillates and does not tend to a limit. On the other hand it looks like $f'(x)/g'(x) \to 0$ as $x \to \infty$. What additional hypothesis on l'Hôpital's rule (mentioned in lectures) is not satisfied here?

5. Suppose $f(x)$ has a Taylor polynomial $T_3(x) = x + \alpha x^2 + \beta x^3$ of order three about $x = 0$. (Do not assume that $f(x)$ has a third derivative anywhere except at $x = 0$, and so no useful remainder terms can be called upon.)

(a) Read off $f(0)$ and the derivatives $f'(0)$, $f''(0)$ and $f'''(0)$.

(b) Prove that $f'(x)$ is positive on an interval covering $x = 0$ and conclude that $f(x)$ has a unique inverse $g(x) = f^{-1}(x)$ on an interval covering $x = 0$.

(c) Differentiate the identity $g(f(x)) = x$ three times and deduce the values of $g(0)$, $g'(0)$, $g''(0)$ and $g'''(0)$.

(d) What is the Taylor polynomial $\tilde{T}_3(x)$ for $f^{-1}(x)$ of order three about $x = 0$?

(e) Read off the Taylor polynomial of order four for $\sin^{-1} x$ about $x = 0$.

6. (a) The function $f(x) = \sin^{-1} x$ has the derivative $f'(x) = (1 - x^2)^{-1/2}$ on the interval $(-1, 1)$. Use the binomial series (not the method in Question **5**) to deduce the Taylor polynomial $T_{2n}(x)$ of order $2n$ for $\sin^{-1} x$ about $x = 0$. (Because the inverse sine is odd, the actual degree of $T_{2n}(x)$ is $2n-1$.) Express the coefficients in terms of binomial coefficients of index $-1/2$ and also in terms of integer factorials.

(b) Do the same for $g(x) = \tan^{-1} x$.

7. (a) Define the function,

$$G(x) = \begin{cases} \cos\sqrt{x}\,, & x \geq 0, \\ \cosh\sqrt{-x}\,, & x < 0. \end{cases}$$

Show that $G(x)$ is differentiable to all orders at $x = 0$ and give an explicit formula for the nth derivative $G^{(n)}(0)$. [*Hint*. Consider the Taylor polynomials of order $2n$ for $\cos x$ and $\cosh x$ about $x = 0$.]

(b) Do the same for the function, $H(x) = \begin{cases} (\sin x)/x, & x \neq 0 \\ 1, & x = 0. \end{cases}$.

(c) In terms of factorials, write down the 50th derivative of $\sin(x^{10})$ at $x = 0$.

8. (a) Show that for a quadratic polynomial, the number c in the Mean Value Theorem (as it is stated in the lecture notes) is always the midpoint of the interval $[a, b]$, $a < b$.

(b) On the other hand, show that c is never the midpoint in the case of a cubic polynomial. Find c in the case when the chord joining the endpoints of the cubic curve segment on $[a, b]$ is tangent to the curve at $x = b$.

(c) In the case of $f(x) = \sin x$ and $g(x) = \cos x$, show that the midpoint of the interval $[a, b]$ is one of the values of c occurring in the Cauchy Mean Value Theorem (lecture notes page 71). How many distinct values of c satisfy the theorem for given $[a, b]$, $a < b$, with $b - a$ not a multiple of 2π.

9. Let $f : [-1, 3] \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 4x^2 + x, & -1 \le x < 0, \\ 2\sqrt{x}\,, & 0 \le x < 1, \\ (4x^3 - 21x^2 + 36x - 7)/6, & 1 \le x \le 3. \end{cases}$$

Find all the critical points of f on $(-1, 3)$, identify each type of critical point (horizontal tangent, vertical tangent, vertical cusp, corner point, other), and decide which critical points are local extrema. Find also the absolute extrema of f on $[-1, 3]$.

10. Draw a set of level curves (corresponding to equally spaced heights) for the following functions:

(a) $f(x, y) = e^{x^2+y^2}$.

(b) $g(x, y) = \dfrac{xy}{x^2 + y^2}$.

(c) $h(x, y) = \left(\sqrt{x} + \sqrt{y}\right)^2$.

Solutions:

1. (a) Denote the eight parts of this exercise (i), (ii), ..., $(viii)$.

(i). The limit,

$$\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^3 - 4x + 3},$$

was done on Sample Quiz 1 by dividing out the comon factor $x - 1$. It can also be done by l'Hôpital's rule for 0/0-type limits:

$$\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^3 - 4x + 3} = \lim_{x\to 1} \frac{3x^2 - 5}{3x^2 - 4} = \left[\frac{3x^2 - 5}{3x^2 - 4}\right]_{x=1} = 2.$$

(ii). Let $L = \lim_{x\to 0^+}(\sinh x)^{1/x}$. The continuity of the logarithm function allows us to write $\ln L = \lim_{x\to 0^+}(\ln \sinh x)/x$ (even though we will end up on the boundary of the domain of continuity). This is not a l'Hôpital problem. The numerator tends to $-\infty$ and the denominator tends to 0^+. Hence, $\ln L = -\infty$ and the required limit is $L = 0$.

Another way to approach this problem is to use the fact that $(\sinh x)/x \to 1$ as $x \to 0$. Then, for some interval $0 < x < \delta$, we have $0 < \sinh x < 2x$ and $0 < (\sinh x)^{1/x} < (2x)^{1/x}$. Since $(2x)^{1/x} \to 0$ as $x \to 0^+$, it follows that $(\sinh x)^{1/x} \to 0$ as $x \to 0^+$.

The required limit can be said to be of 0^∞ type. This is not an indeterminate form, and all such limits are zero.

(iii). Let $L = \lim_{x\to\infty}(\sinh x)^{1/x}$. This can be done with or without l'Hôpital's rule. Take logarithms and use l'Hôpital's rule for ∞/∞-type limits:

$$\ln L = \lim_{x\to\infty} \frac{\ln \sinh x}{x} = \lim_{x\to\infty} \frac{\coth x}{1} = 1,$$

which implies that $L = e$. Alternatively,

$$(\sinh x)^{1/x} = \left(\frac{e^x - e^{-x}}{2}\right)^{1/x} = \frac{e}{2^{1/x}}\left(1 - e^{-2x}\right)^{1/x} \to \frac{e}{1}(1 - 0)^0 = e,$$

as $x \to \infty$. Either way, $L = e$.

(iv). The limit $\lim_{x\to 2}(x^x - 4)/(2^x - 4)$ is of l'Hôpital 0/0 type. We need the derivatives of x^x and 2^x. Use logarithmic differentiation to differentiate variable powers. Let $y = f(x)^{g(x)}$, on a domain where $f(x) > 0$. Then,

$$\ln y = g(x)\ln f(x), \qquad \frac{y'}{y} = g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)},$$

$$\frac{d}{dx} f(x)^{g(x)} = f(x)^{g(x)}\left(g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}\right).$$

In particular,

$$\frac{d}{dx} x^x = x^x(1 + \ln x), \qquad \frac{d}{dx} a^x = (\ln a)a^x, \qquad a > 0.$$

Hence, l'Hôpital's rule gives

$$\lim_{x\to 2} \frac{x^x - 4}{2^x - 4} = \lim_{x\to 2} \frac{x^x(1 + \ln x)}{2^x \ln 2} = \frac{2^2(1 + \ln 2)}{2^2 \ln 2} = \frac{1 + \ln 2}{\ln 2}.$$

(v). Let $L = \lim_{x\to 0^+} x^x$. Then

$$\ln L = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0,$$

which implies that the required limit is $\lim\limits_{x\to 0^+} x^x = 1$.

(vi). Since $(d/dx)x^x = x^x(1+\ln x)$, while $x^x \to 1$ and $\ln x \to -\infty$ as $x \to 0^+$, it follows that the required limit is $\lim_{x\to 0^+}(d/dx)x^x = -\infty$. Thus the graph of $y = x^x$ has a one-sided vertical tangent at $x = 0$, approaching $y = 1$ from below.

(vii). The limit,

$$\lim_{x\to 0} \frac{(ax - \sin ax)^2}{(1-\cos bx)^3},$$

can be evaluated in several ways. L'Hôpital's rule certainly works, but it requires six applications. The quickest way is to use Taylor polynomials of suitable degree. The Taylor polynomial of order three or four of $\sin ax$ is $ax - (ax)^3/3!$. The Taylor polynomial of order two or three of $\cos bx$ is $1 - (bx)^2/2!$. Hence,

$$\lim_{x\to 0} \frac{(ax - \sin ax)^2}{(1-\cos bx)^3} = \lim_{x\to 0} \frac{(a^3x^3/6)^2}{(b^2x^2/2)^3} = \lim_{x\to 0} \frac{a^6x^6/36}{b^6x^6/8} = \frac{2a^6}{9b^6}.$$

Alternatively, one can first obtain the auxiliary limits,

$$L_1 = \lim_{x\to 0} \frac{ax - \sin ax}{x^3} = \frac{a^3}{6}, \qquad L_2 = \lim_{x\to 0} \frac{1-\cos bx}{x^2} = \frac{b^2}{2},$$

which can be done with either Taylor or l'Hôpital (the second limit also appearing on Sample Quiz 1 as a consequence of $(\sin x)/x \to 1$ as $x \to 0$). The required limit is

$$\frac{(L_1)^2}{(L_2)^3} = \frac{a^6/36}{b^6/8} = \frac{2a^6}{9b^6}.$$

$(viii)$. The case $n = 0$ is the limit $e^{-1/x^2} \to 0$ as $x \to 0$ (two-sided). For $n = 1, 2, 3, \ldots$, l'Hôpital's rule works if we recast the given $0/0$-type limit in ∞/∞ form. (If this is not done, l'Hôpital raises the index n and goes the wrong way.) We find

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n} = \lim_{x\to 0} \frac{x^{-n}}{e^{1/x^2}} = \lim_{x\to 0} \frac{-nx^{-n-1}}{-(2/x^3)e^{1/x^2}} = \frac{n}{2} \lim_{x\to 0} \frac{x^{-n+2}}{e^{1/x^2}}.$$

This proves that the limit is zero in the cases $n = 1, 2$. For $n > 2$, use l'Hôpital repeatedly or argue by mathematical induction. Either way,

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n} = 0, \qquad n = 0, 1, 2, 3, \ldots.$$

Another approach is to restrict attention to the right limit $x \to 0^+$ (the left limit being plus or minus the right limit) and let $x = y^{-1/2}$. Then

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^n} = \lim_{y\to\infty} \frac{y^{n/2}}{e^y}.$$

Repeated application of l'Hôpitals rule brings this to the limit of $A_n e^{-y}$ or $B_n y^{-1/2} e^{-y}$ as $y \to \infty$, according as n is even or odd. The final limit reached is obviously zero. A third option is to write

$$\lim_{y\to\infty} \frac{y^{n/2}}{e^y} = \lim_{y\to\infty} \left(\frac{y}{e^{2y/n}}\right)^{n/2}.$$

Now a single application of l'Hôpital's rule and the substitution law for limits gives the limit zero.

This exercise implies that the function $f(x) = e^{-1/x^2}$ with $f(0) = 0$ has vanishing derivatives to all orders at $x = 0$. Hence $f(x)$ has a convergent Taylor series about $x = 0$ that converges to a different function (the zero function).

(b) We have seven comparisons to make between eight sequences that diverge monotonically to $+\infty$ (at least from some term onwards). Suppose $\{a_n\}$ is any sequence of real or complex numbers and $\{b_n\}$ is a monotonic sequence of positive real numbers that tends to 0 or $+\infty$. Use the notation $a_n << b_n$ and $b_n >> a_n$ to denote the fact that $|a_n|/b_n \to 0$ as $n \to \infty$. When both sequences are real and diverge to $+\infty$, it is useful to know that $a_n << b_n$ implies $e^{a_n} << e^{b_n}$ and vice versa.

(i). $n^{100} = e^{100 \ln n}$ and $n^{\ln n} = e^{(\ln n)^2}$. Since $(\ln n)^2 >> 100 \ln n$, it follows that $n^{\ln n} >> n^{100}$. Similarly $n^{\ln n} >> n^k$ for every fixed $k > 0$ (and trivially for $k \le 0$).

(ii). L'Hôpital's rule implies that $\sqrt{n} >> (\ln n)^2$. Hence, $e^{\sqrt{n}} >> n^{\ln n}$.

(iii). $n^{\sqrt{n}} = e^{\sqrt{n}\ln n}$. Since $\sqrt{n} \ln n >> \sqrt{n}$, it follows that $n^{\sqrt{n}} >> e^{\sqrt{n}}$.

(iv). $2^n = e^{n \ln 2}$. L'Hôpital's rule implies that $n \ln 2 >> \sqrt{n} \ln n$. Hence $2^n >> n^{\sqrt{n}}$.

(v). We will show that $n! >> a^n$ for every real a. Without loss of generality, we can take $a > 1$. Let m be a fixed integer $> 2a$ and let $n > m$. Then

$$\begin{aligned}
\frac{a^n}{n!} &= \frac{a \cdot a \cdot a \cdots a}{1 \cdot 2 \cdot 3 \cdots n} \\
&= \frac{a^m}{m!}\left(\frac{a}{m+1}\right)\left(\frac{a}{m+2}\right)\cdots\left(\frac{a}{n}\right) \\
&< \frac{a^m}{m!}\left(\frac{1}{2}\right)^{n-m} \\
&= \frac{A_m}{2^n},
\end{aligned}$$

where A_m is a constant depending on a and m. Since $A_m/2^n \to 0$ as $n \to \infty$, it follows that $a^n/n! \to 0$ as $n \to \infty$. Thus $n! >> a^n$. In particular $n! >> 2^n$.

(vi).

$$\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} < \frac{1}{n}.$$

Hence $n^n >> n!$. (More generally, $n^n >> a^n\, n!$ for $0 < a < e$ and $n^n << a^n\, n!$ for $a \ge e$.)

(vii). $n^n = e^{n \ln n}$ and $2^{n^2} = e^{n^2 \ln 2}$. Since $n^2 \ln 2 >> n \ln n$, it follows that $2^{n^2} >> n^n$.

2. The derivatives of $\sin x$ and $\cos x$ are

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \sin x\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right) + \cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right), \\
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \cos x\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right) - \sin x\left(\lim_{h\to 0}\frac{\sin h}{h}\right).
\end{aligned}$$

The addition theorems for $\sin(x+h)$ and $\cos(x+h)$ are geometric in origin and do not depend on calculus or on complex exponentials. [Exercise: derive the addition theorems from $\sin\theta = \text{opp/hyp}$ and $\cos\theta = \text{adj/hyp}$.] The two-sided limit,

$$\lim_{h\to 0}\frac{\sin h}{h} = 1,$$

was proved in lectures by a geometric argument. (It is also proved in the lecture notes, pages 43–44, by a different geometric argument.) In particular, we showed that $\sin h < h < \tan h$ for $0 < h < \pi/2$. An exercise on Sample Quiz 1 included the result,

$$\lim_{h\to 0}\frac{1-\cos h}{h^2} = \lim_{h\to 0}\frac{1+\cos h}{2}\lim_{h\to 0}\frac{1-\cos h}{h^2} = \lim_{h\to 0}\frac{1-\cos^2 h}{2h^2} = \lim_{h\to 0}\frac{\sin^2 h}{2h^2} = \frac{1}{2}.$$

Multiplying by $-h$ gives the limit that we want here:

$$\lim_{h\to 0}\frac{\cos h-1}{h} = 0.$$

Note that it is logically incorrect to use l'Hôpital's rule in these instances, as it assumes results that we are trying to prove. We now have the two auxiliary limits that we need to complete the proof of

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x.$$

See the appendix below for a first-principles evaluation of $(d/dx)\log_b x$.

3. The seven statements are, in turn, False, True, True, False, False, True, False. The false statements can be dismissed quickly with a counerexample. Let $g_a(x)$, $a > 0$, denote the function $g_a(x) = |x|^a\sin(1/x)$, with $g_a(0) = 0$, which is continuous everywhere. We are given that $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$.

(a) "$f(x)$ must have a right derivative at $x = 0$." This is false because the graph of $y = x^a$, $0 < a < 1$, on $[0,1]$ has a one-sided vertical tangent at the left endpoint $x = 0$. In MATH1901, we do not count infinite derivatives as valid derivatives, but some analysis texts do allow them. So look instead at the function $g_a(x)$ with $0 < a \le 1$, whose difference quotient $(g_a(x) - g_a(0))/x$ behaves rather badly in the limit $x \to 0^+$.

(b) "If $f'(x)$ tends to a finite limit L as $x \to 0^+$, then $f(x)$ necessarily has a right derivative at $x = 0$ whose value is L." This is true and is a consequence of the Mean Value Theorem, whose conditions are satisfied here. On the interval $[0, \delta]$, the MVT states that

$$\frac{f(x) - f(0)}{x} = f'(c)$$

for some c such that $0 < c < x$. If $x \to 0^+$, then c is forced to tend to 0^+ as well (possibly in irregular steps, skipping values). Hence, if $f'(x)$ tends to a limit L as $x \to 0^+$, then $f'(c)$ will tend to the same limit and the difference quotient $(f(x) - f(0))/x$ will also tend to the same limit. In other words, the right derivative $f'_+(0)$ exists and equals L.

(c) "If $f'(x)$ tends to either $+\infty$ or $-\infty$ as $x \to 0^+$, then the graph of $y = f(x)$ on $[0,1]$ must have a one-sided vertical tangent at $x = 0$." This is also true and is a similar consequence of the MVT. If $f'(x) \to +\infty$ as $x \to 0^+$, the difference quotient $(f(x) - f(0))/x$ will also tend to $+\infty$ as $x \to 0^+$. This is precisely what identifies a tangent ray pointing vertically up at the left endpoint $x = 0$ on the graph of $y = f(x)$. Similarly $f'(x) \to -\infty$ as $x \to 0^+$ implies that the graph of $y = f(x)$ has a tangent ray pointing vertically down at $x = 0$. The two cases are illustrated by $f(x) = x^a$ and $f(x) = -x^a$ with $0 < a < 1$.

(d) "If $f'(x)$ does not tend to a finite limit as $x \to 0^+$, then $f(x)$ cannot have a right derivative at $x = 0$." This is false, though perhaps not obviously so. The function $g_a(x)$ with $1 < a \le 2$ provides a counterexample. When $x > 0$, the product and chain rules give

$$g_a'(x) = \frac{d}{dx} x^a \sin(1/x) = a x^{a-1} \sin(1/x) - x^{a-2} \cos(1/x).$$

The second term has unbounded oscillations as $x \to 0^+$ when $1 < a < 2$. When $a = 2$, the oscillations are bounded. Either way, $g_a'(x)$ does not tend to a limit as $x \to 0^+$. However, $g_a'(0)$ exists as a two-sided derivative for all $a > 1$ because

$$\begin{aligned} g_a'(0) &= \lim_{x \to 0} \frac{g_a(x) - g_a(0)}{x} \\ &= \lim_{x \to 0} \frac{|x|^a \sin(1/x) - 0}{x} \\ &= \lim_{x \to 0} \pm |x|^{a-1} \sin(1/x) \\ &= 0. \end{aligned}$$

The last step is justified by the squeeze lemma when $a > 1$.

This example shows that $f'(x)$ can exist on an interval and have an isolated discontinuity at one point, say x_0. However, regardless of whether or not $f'(x_0)$ exists, part (b) shows that the discontinuity in $f'(x)$ cannot be a removable discontinuity.

(e) "If $f(x)$ satisfies the inequality $ax \le f(x) \le bx$ on $[0, 1]$ for particular real numbers a, b, $a < b$, then $f'(x)$ satisfies $a \le f'(x) \le b$ on $(0, 1)$." This is false because a differentiable function can have rapid oscillations with small amplitudes. The given bounds on $f(x)$ are satisfied by

$$f(x) = \tfrac{1}{2}(b + a)x + \tfrac{1}{2}(b - a)g_c(x), \qquad c \ge 1.$$

However, $f'(x)$ has unbounded oscillations when $1 \le c < 2$ and therefore cannot be contained between a and b.

(f) "There exists a point $c \in (0, 1)$ such that $f'(c) = f(1) - f(0)$." This is true because it is precisely what the MVT states for $f(x)$ on $[0, 1]$.

(g) "The point c in the previous part is unique when it exists." This is false because a differentiable function $f(x)$ can have several turning points or changes of concavity on $(0, 1)$. If any line with slope $f(1) - f(0)$ cuts the graph three or more times, then the MVT on each subinterval provides a distinct value of c. So c is only unique in special circumstances.

4. (a) A direct proof that the graph of $y = f(x) = x^{1/3}$ has a vertical tangent at $x = 0$ can be provided by the difference quotient:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{x^{1/3}}{x} = \lim_{x\to 0} x^{-2/3} = +\infty.$$

The limit is two-sided, and so the vertical tangent is two-sided and the curve is smooth there (in contrast to the case of a vertical cusp).

In view of the statement in Question **3**(c), a vertical tangent can be identified more quickly by the following two-sided limit:

$$\frac{d}{dx}\, x^{1/3} = \frac{1}{3}\, x^{-2/3} \to +\infty,$$

as $x \to 0$. (Of course, not all vertical tangents can be captured this way. Consider the case $f(x) = x^{1/3} + x^{6/5} \sin(1/x)$.)

To prove that $x = 0$ is also a point of inflection of $x^{1/3}$, consider $f''(x)$ on either side. For $x \ne 0$,

$$f'(x) = \frac{1}{3}\, x^{-2/3}, \qquad f''(x) = -\frac{2}{9}\, x^{-5/3}.$$

Thus $f''(x) < 0$ for $x > 0$ and $f''(x) > 0$ for $x < 0$. Thus $f(x)$ changes concavity from up to down at $x = 0$, which is precisely what identifies $x = 0$ as a point of inflection.

Another way to view this problem is to observe that $f : \mathbf{R} \to \mathbf{R}$, $x \mapsto x^{1/3}$, is bijective and has an inverse function $f^{-1}(x) = x^3$. The horizontal tangent and inflection of x^3 at $x = 0$ implies the vertical tangent and inflection of $x^{1/3}$ at $x = 0$ and vice-versa.

(b) We are given that $f'(x_0) = 0$ and $f''(x_0) > 0$. If $f''(x)$ is continuous at x_0, then $f''(x) > 0$ on an interval covering x_0. That would imply that $f(x)$ is concave up, and therefore has a local minimum at a point where the tangent is horizontal. The question asked us not to assume that $f''(x)$ even exists away from x_0. Nevertheless, several conclusions can be drawn from the existence of $f''(x)$ at just one point x_0. The definition of derivative requires that $f'(x)$ be defined on an interval covering x_0 and be continuous at x_0 itself. Then $f(x)$ must be continuous on an interval covering x_0. Next,

$$f''(x_0) = \lim_{x\to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{f'(x)}{x - x_0}.$$

Given arbitrary $\epsilon > 0$, there exists $\delta > 0$ (depending on ϵ) such that

$$\left| \frac{f'(x)}{x - x_0} - f''(x_0) \right| < \epsilon$$

for all x such that $0 < |x - x_0| < \delta$. For such x,

$$f''(x_0) - \epsilon < \frac{f'(x)}{x - x_0} < f''(x_0) + \epsilon.$$

Since $f''(x_0) > 0$, we can choose ϵ anywhere in the interval $0 < \epsilon < f''(x_0)/2$. Then

$$\frac{f'(x)}{x - x_0} > \frac{f''(x_0)}{2}$$

for all x such that $0 < |x - x_0| < \delta$. This result implies

$$f'(x) > \frac{f''(x_0)}{2}(x - x_0) > 0, \qquad x_0 < x < x_0 + \delta,$$

$$f'(x) < \frac{f''(x_0)}{2}(x - x_0) < 0, \qquad x_0 - \delta < x < x_0.$$

These inequalities imply that $f(x)$ is strictly increasing for $x > x_0$ near x_0 and strictly decreasing for $x < x_0$ near x_0. Hence, we have proved that $f(x)$ has a strict local minimum at x_0 under the stated hypotheses. (In case you were wondering what a non-strict local minimum looks like, consider $x^2 \sin^2(1/x)$ near $x = 0$.)

(c) The corner point in the graph of $y = f(x) = \sin^{-1}(1 - x^2)$ can be seen with or without taking a derivative. If we do it with derivatives, we need to know that

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}, \qquad -1 < x < 1,$$

which can be proved by differentiating the identity $\sin(\sin^{-1} x) = x$ with the chain rule. Then, for $x \neq 0$, the chain rule gives

$$\begin{aligned} f'(x) &= \frac{d}{dx} \sin^{-1}(1 - x^2) \\ &= -\frac{2x}{\sqrt{1 - (1 - x^2)^2}} \\ &= -\frac{2x}{\sqrt{2x^2 - x^4}} \\ &= -\frac{2x}{|x|\sqrt{2 - x^2}}. \end{aligned}$$

Since $f'(x)$ has one-sided limits as $x \to 0^{\pm}$, these are the one-sided derivatives at $x = 0$, according to Question **3**(b). So we get the right and left derivatives,

$$f'_{+}(0) = -\sqrt{2}\,, \qquad f'_{-}(0) = \sqrt{2}\,.$$

Non-equal one-sided derivatives imply that the graph has a corner point at $x = 0$. It is also a local and absolute maximum (which is obvious from the original function). More generally, any continuous even function with a nonzero right derivative at $x = 0$ has a left derivative of opposite sign, and therefore a corner point at $x = 0$.

To see the corner point without taking derivatives, we exploit the limit $(\sin x)/x \to 1$ as $x \to 0$. Use the notation $f(x) \sim g(x)$ as $x \to x_0$ to mean that $f(x)/g(x) \to 1$ as $x \to x_0$. In particular, $\sin x \sim x$ as $x \to 0$, which implies that $\sin^{-1} x \sim x$ as $x \to 0$. We need to express the given inverse sine in terms of an inverse sine of small argument. Apply the trigonometric identity,

$$\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x = \frac{\pi}{2} - \sin^{-1}\sqrt{1-x^2}\,,$$

which is valid for $0 \le x \le 1$. We find, for $-1 < x < 1$,

$$\begin{aligned}
\sin^{-1}(1-x^2) &= \frac{\pi}{2} - \cos^{-1}(1-x^2) \\
&= \frac{\pi}{2} - \sin^{-1}\sqrt{1-(1-x^2)^2} \\
&= \frac{\pi}{2} - \sin^{-1}\sqrt{2x^2 - x^4} \\
&= \frac{\pi}{2} - \sin^{-1}\big(|x|\sqrt{2-x^2}\,\big) \\
&\sim \frac{\pi}{2} - |x|\sqrt{2-x^2} \qquad \text{as } x \to 0 \\
&\sim \frac{\pi}{2} - \sqrt{2}\,|x| \qquad \text{as } x \to 0.
\end{aligned}$$

Thus $\sin^{-1}(1-x^2)$ has a corner point of the same character as the absolute value function (upside-down because of the minus sign, steeper on account of the factor $\sqrt{2}$, which is the left derivative at $x = 0$).

(d) We are given the function,

$$f(x) = \begin{cases} |x|^a \sin(1/|x|^b), & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad a > 0, \quad b > 0,$$

which is even and continuous at $x = 0$. Before we can examine second derivatives, we need to know when $f(x)$ has a first derivative at $x = 0$. Consider the difference quotient,

$$\frac{f(x) - f(0)}{x} = \frac{|x|^a}{x}\sin(1/|x|^b) = \pm\,|x|^{a-1}\sin(1/|x|^b).$$

Since this is squeezed between $|x|^{a-1}$ and $-|x|^{a-1}$, the limit $x \to 0$ exists and equals zero whenever $a > 1$. The limit does not exist when $a \le 1$. Thus

$$f'(0) = 0, \qquad a > 1,$$

and $f'(0)$ does not exist for $a \leq 1$. To find when $f'(x)$ is continuous at $x = 0$, apply the product and chain rules when $x > 0$:

$$f'(x) \;=\; \frac{d}{dx} x^a \sin(x^{-b}) \;=\; a x^{a-1} \sin(x^{-b}) - b x^{a-b-1} \cos(x^{-b}).$$

The evenness of $f(x)$ implies $f'(-x) = -f'(x)$. So $f'(x) \to 0$ when $x \to 0$ (two-sided) whenever $a > b+1$, while $f(x)$ oscillates as $x \to 0$ if $a \leq b+1$. Hence $f'(x)$ is continuous at $x = 0$ whenever $a > b+1$.

To get the second derivative $f''(0)$, assume that $a > b+1$ and consider, for $x > 0$, the difference quotient,

$$\frac{f'(x) - f'(0)}{x} \;=\; a x^{a-2} \sin(x^{-b}) - b x^{a-b-2} \cos(x^{-b}).$$

This tends to zero whenever $a > b+2$ and does not tend to a limit when $a \leq b+2$. Because $f''(x)$ is even, the limit from the left is the same. Hence,

$$f''(0) \;=\; 0, \qquad a > b+2,$$

and $f''(0)$ does not exist for $a \leq b+2$. To find when $f''(x)$ is continuous at $x = 0$, apply the product and chain rules as before with $x > 0$:

$$\begin{aligned} f''(x) \;&=\; \frac{d}{dx}\big\{a x^{a-1} \sin(x^{-b}) - b x^{a-b-1} \cos(x^{-b})\big\} \\ &=\; a(a-1)x^{a-2}\sin(x^{-b}) - b(2a-b-1)x^{a-b-2}\cos(x^{-b}) - b^2 x^{a-2b-2}\sin(x^{-b}). \end{aligned}$$

This tends to zero whenever $a > 2b+2$ and does not tend to a limit when $a \leq 2b+2$. We conclude that $f''(0)$ exists if and only if $a > b+2$ and $f''(x)$ is continuous at $x = 0$ if and only if $a > 2b+2$.

(e) Let $f(x) = (1 + 1/x)^x$ and $g(x) = \ln f(x) = x \ln(1 + 1/x)$ for $x > 0$. Then $f(x)$ is increasing if and only if $g(x)$ is increasing. Differentiate $g(x)$:

$$\begin{aligned} g'(x) \;&=\; \frac{d}{dx} x \ln\Big(1 + \frac{1}{x}\Big) \\ &=\; \frac{d}{dx} x\big\{\ln(1+x) - \ln x\big\} \\ &=\; \ln(1+x) - \ln x + x\Big(\frac{1}{1+x} - \frac{1}{x}\Big) \\ &=\; \ln(1+x) - \ln x - \frac{1}{1+x}\,. \end{aligned}$$

Observe that $g'(x) \to 0$ as $x \to \infty$. Since the sign of $g'(x)$ is not immediately obvious, take another derivative:

$$g''(x) \;=\; \frac{1}{1+x} - \frac{1}{x} + \frac{1}{(1+x)^2} \;=\; -\,\frac{1}{x(x+1)^2}\,.$$

Hence $g''(x)$ is negative and $g'(x)$ is decreasing. Since $g'(x)$ decreases to zero in the limit $x \to \infty$, it follows that $g'(x)$ is positive for all $x > 0$. Thence $g(x)$ is increasing and therefore also $f(x)$ is increasing for all $x > 0$. This completes the proof. (L'Hôpital's rule shows that $g(x) \to 1$ and $f(x) \to e$ as $x \to \infty$.)

The appendix below gives a first-principles proof that $(1 + 1/n)^n$ is increasing as n runs through the positive integers, and that the limit is the sum of the rapidly convergent series $\sum_{k=0}^{\infty}(1/k!)$, which is the definition of e in that context.

(f) This exercise illustrates a subtle point in the statement of l'Hôpital's rule. For convenience, we pick the version that applies to ∞/∞-type limits as $x \to \infty$, but a similar point applies to all the versions of l'Hôpital's rule. The brief statement of l'Hôpital's rule in this instance is

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

whenever the limit on the right-hand side exists and $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$. The existence of the limit on the right implies the existence of the limit on the left. (The converse is not true, as in the case $f(x) = x + \sin x$ and $g(x) = x$, and is not expected to be true when one looks at the proof of l'Hôpital's rule.)

Certain general conclusions can be drawn from the existence of the limit on the right. First, there needs to be a neighbourhood of infinity, in other words, an interval (a, ∞), on which $g'(x)$ is nonzero. That implies in turn that there is a neighbourhood of infinity on which $g(x)$ is nonzero, for otherwise a sequence of values of x on which $g(x) = 0$ would imply a sequence on which $g'(x) = 0$ by Rolle's theorem. That is why no separate hypothesis about $g(x)$ being nonzero in the neighbourhood of the limiting point is needed in the statement of l'Hôpital's rule.

But perhaps we could allow $g'(x)$ to have zeros that are cancelled by corresponding zeros in $f'(x)$. Then $f'(x)/g'(x)$ would have removable discontinuities that we could choose to remove automatically while taking the limit. Can this be allowed? Consider the given case,

$$f(x) = x + \sin x \cos x, \qquad g(x) = e^{\sin x}(x + \sin x \cos x).$$

Then $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$, but $f(x)/g(x) = e^{-\sin x}$ oscillates between e and $1/e$ and does not tend to a limit. According to l'Hôpital's rule, $f'(x)/g'(x)$ should not tend to a limit either. But

$$f'(x) = 2\cos^2 x, \qquad g'(x) = 2e^{\sin x}\cos^2 x + e^{\sin x}\cos x(x + \sin x \cos x).$$

$f'(x)$ and $g'(x)$ have the common factor $\cos x$. If we cancel that factor,

$$\frac{f'(x)}{g'(x)} = \frac{2e^{-\sin x}\cos x}{x + \sin x \cos x + 2\cos x}.$$

The numerator oscillates between $-2e$ and $2e$ (not attaining these bounds). The denominator is strictly positive for $x > 3$ and tends to $+\infty$ as $x \to \infty$. Hence, $f'(x)/g'(x)$ tends to the limit zero as $x \to \infty$, and we have found an apparent counterexample to l'Hôpital's rule.

Recall that we cancelled a common factor $\cos x$ in $f'(x)$ and $g'(x)$. This factor vanishes in every neighbourhood of infinity, and so to get the limit of $f'(x)/g'(x)$ we had to agree to automatically remove removable discontinuities as they occur. So by allowing this possibility, we ended up with a counterexample to L'Hôpitals's rule. That is why, in the lectures, to disallow these sorts of counterexamples, we added an extra hypothesis to the statement of l'Hôpital's rule to the effect that $g'(x)$ is nonzero in some deleted neighbourhood of the limiting point (where "deleted" means that the limiting point itself is cut out of the neighbourhood). In MATH1901, we take the view that removable discontinuities are not included in the domain of a function unless we give a separate definition of the function at such points. But even with that understanding, we could still make the case that $f'(x)/g'(x)$ in the example above tends to zero as $x \to \infty$ through the domain on which it is defined. The extra hypothesis is not needed if neither work-around is tolerated, in other words, if the existence of the limit of $f'(x)/g'(x)$ in l'Hôpital's rule is understood to imply that $g'(x) \neq 0$ in some deleted neighbourhood of the limiting point regardless of what $f'(x)$ does.

5. We are given that $f(x)$ admits the Taylor polynomial of order three $T_3(x) = x + \alpha x^2 + \beta x^3$ about $x = 0$.

(a) To form $T_3(x)$ about $x = 0$, the derivatives $f'(0)$, $f''(0)$ and $f'''(0)$ must exist. We are told not to assume that $f'''(x)$ exists anywhere else. However, in order for $f'''(0)$ to exist, the lower derivatives must exist on an interval covering $x = 0$. The formula for $T_3(x)$ is

$$T_3(x) \;=\; f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3.$$

Using $2! = 2$ and $3! = 6$, we read off

$$f(0) = 0, \qquad f'(0) = 1, \qquad f''(0) = 2\alpha, \qquad f'''(0) = 6\beta.$$

(b) The existence of the limit $(f''(x) - f''(0))/x \to f'''(0)$ as $x \to 0$ implies that there is an interval $(-\delta, \delta)$ covering $x = 0$ on which $f''(x)$ is both defined and bounded. Then $f'(x)$ is continuous and differentiable on $(-\delta, \delta)$. Hence $f'(t)$ satisfies the conditions of the Mean Value Theorem on intervals $0 < t < x$ for the case $0 < x < \delta$ and on intervals $x < t < 0$ for the case $-\delta < x < 0$. The MVT states that

$$f'(x) \;=\; f'(0) + f''(c)x,$$

for some c between 0 and x. By choosing a smaller δ, if necessary, we can guarantee that $|f''(x)| < A$ and $|x| < 1/(2A)$ on the interval $(-\delta, \delta)$ for some positive constant A. Since $f'(0) = 1$, we have found an interval covering $x = 0$ on which $f'(x) > 1/2$. On that interval, $f(x)$ is strictly increasing and one-to-one. It therefore has a unique functional inverse $g(x) = f^{-1}(x)$ near $x = f(0)$. Since $f(0) = 0$, we see that $g(0) = 0$ and $g(x)$ is also defined and strictly increasing on an interval covering $x = 0$.

(c) The inverse function $g(x)$ inherits its differentiability properties from $f(x)$ through the identities $g(f(x)) = x$ and $f(g(x)) = x$, valid near $x = 0$. Taking three derivatives of the first identity using the chain rule gives

$$\begin{aligned} g(f(x)) &= x, \\ f'(x)g'(f(x)) &= 1, \\ f''(x)g'(f(x)) + \{f'(x)\}^2 g''(f(x)) &= 0, \\ f'''(x)g'(f(x)) + 3f'(x)f''(x)g''(f(x)) + \{f'(x)\}^3 g'''(f(x)) &= 0. \end{aligned}$$

The last derivative may only be possible at $x = 0$. The identity $f(g(x)) = x$ produces the same results with f and g swapped. Either way, we get

$$g(0) = 0, \qquad g'(0) = 1, \qquad g''(0) = -f''(0) = -2\alpha,$$

$$g'''(0) = -3f''(0)g''(0) - f'''(0) = 12\alpha^2 - 6\beta.$$

(d) From the previous result, we can read off the Taylor polynomial,

$$\begin{aligned} \tilde{T}_3(x) &= g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 \\ &= x - \alpha x^2 + (2\alpha^2 - \beta)x^3, \end{aligned}$$

of order three for $g(x) = f^{-1}(x)$. (We placed a tilde on the T to avoid confusion with the Taylor polynomial for $f(x)$.)

(e) In the special case $f(x) = \sin x$, for which

$$T_3(x) = T_4(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6},$$

about $x = 0$, we read off

$$\tilde{T}_3(x) = x + \frac{x^3}{6}$$

for the inverse function $g(x) = \sin^{-1} x$. But since $\sin^{-1} x$ is odd and differentiable to all orders on $(-1, 1)$, we know that $g^{(4)}(0) = 0$. Hence,

$$\tilde{T}_4(x) = x + \frac{x^3}{6}$$

for $\sin^{-1} x$ about $x = 0$.

6. (a) To get the complete Taylor series, or, equivalently, the Taylor polynomial of any order, for $\sin^{-1} x$ about $x = 0$, we use the facts that

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}, \qquad -1 < x < 1,$$

and that the right-hand side has a binomial series expansion. The inverse trig functions are bijective on the following domains and codomains:

$$\begin{aligned} \sin^{-1} : [-1, 1] &\to [-\pi/2, \pi/2], \\ \cos^{-1} : [-1, 1] &\to [0, \pi], \\ \tan^{-1} : \mathbf{R} &\to (-\pi/2, \pi/2). \end{aligned}$$

To get the derivative of $\sin^{-1} x$, let $y = \sin^{-1} x$. Then $x = \sin y$ with domain $-\pi/2 \le y \le \pi/2$. Its derivative is $dx/dy = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$, where the nonnegative square root is the correct one. Taking reciprocals to get dy/dx and removing the endpoints $x = \pm 1$ from the x-domain where the tangents are vertical, we get $dy/dx = (d/dx)\sin^{-1} x = (1-x^2)^{-1/2}$. [Similarly, $(d/dx)\cos^{-1} x = (d/dx)(\pi/2 - \sin^{-1} x) = -(1-x^2)^{-1/2}$ and $(d/dx)\tan^{-1} x = 1/(1+x^2)$.]

The binomial series for $g(x) = (1+x)^\alpha$, $\alpha \in \mathbf{R}$, is its Taylor series about $x = 0$. We find

$$g(0) = 1, \qquad g'(0) = \alpha, \qquad g''(0) = \alpha(\alpha-1),$$

and, in the general case,

$$g^{(k)}(0) \;=\; \alpha(\alpha-1)(\alpha-2)\,\cdots\,(\alpha-k+1).$$

The Taylor coefficient $g^{(k)}(0)/k!$ is the binomial coefficient,

$$\frac{g^{(k)}(0)}{k!} \;=\; \binom{\alpha}{k} \;=\; \frac{\alpha(\alpha-1)(\alpha-2)\,\cdots\,(\alpha-k+1)}{k!}\,,$$

which is 1 when $k = 0$. The binomial series is

$$(1+x)^\alpha \;=\; \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k,$$

valid at least for $-1 < x < 1$. [*Remarks.* This domain of validity was stated in lectures but not proved. The proof appears as a tutorial exercise in MATH1903. The standard Taylor remainder term works well in the smaller domain $-1/2 < x < 1$. The endpoint -1 is included when $\alpha \ge 0$. The endpoint 1 is included when $\alpha > -1$. When α is a nonnegative integer n, the binomial series terminates as a polynomial of degree n and agrees with the usual binomial theorem of elementary algebra, in which case its domain of validity is $\mathbf{R}$.]

To get the binomial series for $(1-x^2)^{-1/2}$, let $\alpha = -1/2$ and replace x by $-x^2$. We arrive at

$$(1-x^2)^{-1/2} \;=\; \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} x^{2k}, \qquad -1 < x < 1.$$

Since we have not treated the Taylor remainder term for the binomial series properly in lectures, we will have to assume that it is valid to integrate this series term by term. (It is always valid inside the interval of convergence of a power series.) Renaming x to t and integrating both sides with respect to t from 0 to x, we get the Taylor series,

$$\sin^{-1} x \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \binom{-1/2}{k} x^{2k+1}, \qquad -1 \le x \le 1.$$

Interestingly, the validity of this result extends to the endpoints $x = \pm 1$, where the original binomial series diverged to $+\infty$, but the proof will not be given here.

The question asked for the Taylor polynomial of order $2n$, which will be of degree $2n-1$ since only odd powers appear. Truncating the Taylor series at order $2n$ gives

$$\begin{aligned} T_{2n}(x) &= \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{-1/2}{k} x^{2k+1} \\ &= x - \frac{1}{3}\binom{-1/2}{1} x^3 + \frac{1}{5}\binom{-1/2}{2} x^5 - \frac{1}{7}\binom{-1/2}{3} x^7 \\ &\qquad + \ldots + \frac{(-1)^{n-1}}{2n-1} \binom{-1/2}{n-1} x^{2n-1}, \end{aligned}$$

for $f(x) = \sin^{-1} x$ about $x = 0$. The binomial coefficient with index $-1/2$ can be expressed in terms of integer factorials:

$$\begin{aligned} \binom{-1/2}{k} &= \frac{1}{k!}\Big(-\frac{1}{2}\Big)\Big(-\frac{3}{2}\Big)\Big(-\frac{5}{2}\Big) \cdots \Big(-\frac{2k-1}{2}\Big) \\ &= \frac{(-1)^k}{k!} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k} \\ &= \frac{(-1)^k}{2^k\, k!} \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k-1)(2k)}{2 \cdot 4 \cdot 6 \cdots (2k)} \\ &= \frac{(-1)^k}{2^k\, k!} \frac{(2k)!}{2^k\, k!} \\ &= (-1)^k \frac{(2k)!}{2^{2k} (k!)^2}\,. \end{aligned}$$

This can also be written $(-1)^k (2k-1)!/\{2^{2k-1}\, k!(k-1)!\}$. Hence, the Taylor polynomial for $\sin^{-1} x$ about $x = 0$ becomes

$$T_{2n}(x) = \sum_{k=0}^{n-1} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} x^{2k+1}.$$

All the coefficients are positive. The first five nonzero terms are

$$\sin^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \ldots.$$

(b) The corresponding treatment of the inverse tangent is simpler, and it is possible to give a simple remainder term (not the standard Taylor remainder term). First, let $y = \tan^{-1} x$ with domain $\mathbf{R}$ and range $(-\pi/2, \pi/2)$. Then $x = \tan y$, $dx/dy = \sec^2 y = 1 + \tan^2 y = 1 + x^2$. This shows that

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}, \qquad x \in \mathbf{R}.$$

The Taylor polynomials for $1/(1+x)$ are just finite geometric series that we can sum exactly with a well-known formula, and letting $x \to x^2$ means that we can do the same with $1/(1+x^2)$. In the finite geometric series,

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}, \qquad r \neq 1,$$

let $a = 1$ and $r = -x^2$. We find

$$1 - x^2 + x^4 - x^6 + \ldots + (-1)^{n-1}x^{2n-2} = \frac{1 + (-1)^{n-1}x^{2n}}{1 + x^2},$$

valid for all $x \in \mathbf{R}$. The polynomial on the left is the Taylor polynomial or order $2n - 1$ for $1/(1 + x^2)$ about $x = 0$. We can rearrange the previous result as

$$\begin{aligned}
\frac{1}{1 + x^2} &= \tilde{T}_{2n-1}(x) + \tilde{R}_{2n-1}(x), \\
\tilde{T}_{2n-1}(x) &= 1 - x^2 + x^4 - x^6 + \ldots + (-1)^{n-1}x^{2n-2}, \\
\tilde{R}_{2n-1}(x) &= (-1)^n \frac{x^{2n}}{1 + x^2},
\end{aligned}$$

valid for all $x \in \mathbf{R}$. (We placed a tilde on T and R because we wish to use these symbols again for the inverse tangent.) If we wish to form the Taylor series for $1/(1 + x^2)$ about $x = 0$, then we need to restrict x to the open interval $(-1, 1)$ so that the factor x^{2n} in the remainder term tends to zero as $n \to \infty$.

Term-by-term integration of both sides gives

$$\begin{aligned}
\tan^{-1} x &= T_{2n}(x) + R_{2n}(x), \\
T_{2n}(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n - 1}, \\
R_{2n}(x) &= (-1)^n \int_0^x \frac{t^{2n}}{1 + t^2}\, dt,
\end{aligned}$$

valid for all $x \in \mathbf{R}$. Here, $T_{2n}(x)$ is the required Taylor polynomial of order $2n$ for $\tan^{-1} x$ about $x = 0$. It is a polynomial of degree $2n - 1$.

We can stop here since we have answered the question, but we should but a bound on the remainder term and determine the domain of validity of the Taylor series while we are in this position. This topic belongs to the integral theory of Taylor series covered in MATH1903 (we do the differential theory).

The remainder $R_{2n}(x)$ is an odd function of x and has the sign $(-1)^n \operatorname{sgn}(x)$, where $\operatorname{sgn} x$ denotes the sign of x when $x \neq 0$. Since $1/(1 + t^2) \leq 1$, with equality only when $t = 0$, we have, for $x \neq 0$,

$$0 < (-1)^n \operatorname{sgn}(x)\, R_{2n}(x) < \int_0^{|x|} t^{2n}\, dt = \frac{|x|^{2n+1}}{2n + 1}.$$

If we are not interested in the precise sign of the remainder, we can just write

$$0 \leq \bigl|R_{2n}(x)\bigr| \leq \frac{|x|^{2n+1}}{2n + 1},$$

valid for all $x \in \mathbf{R}$, with equality only when $x = 0$. We see that the remainder term tends to zero as $n \to \infty$ whenever x is in the closed interval $[-1, 1]$. The endpoints are included, even though they were not included in the Taylor series for the derivative $1/(1+x^2)$. So we have proved the Taylor series expansion,

$$\tan^{-1} x \;=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\,, \qquad -1 \le x \le 1.$$

Up until the age of computers, this inverse tangent series was the most popular for high-precision calculations of π. It converges well for small x. John Machin (1706) gave the formula,

$$\frac{\pi}{4} \;=\; 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}\,.$$

Numerous improvements to Machin's formula have been provided since then. Even in recent times, when extremely rapidly converging schemes have been developed to calculate π to billions of decimal places, the inverse tangent series is still a serious contender because it is so easy to implement. The current record of 1.24 trillion decimal places set by Yasumasa Kanada and coworkers in 2002 was achieved with the inverse tangent series and the following two Machin-like formulae:

$$\frac{\pi}{4} \;=\; 12\tan^{-1}\frac{1}{49} + 32\tan^{-1}\frac{1}{57} - 5\tan^{-1}\frac{1}{239} + 12\tan^{-1}\frac{1}{110443}\,,$$

$$\frac{\pi}{4} \;=\; 44\tan^{-1}\frac{1}{57} + 7\tan^{-1}\frac{1}{239} - 12\tan^{-1}\frac{1}{682} + 24\tan^{-1}\frac{1}{12943}\,.$$

7. (a) Let $U_{2n}(x)$ and $V_{2n}(x)$ denote the Taylor polynomials of order $2n$ or $2n+1$ about $x=0$ for $\cos x$ and $\cosh x$, respectively. The standard results are

$$U_{2n}(x)\;\;(\text{for } \cos x) \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\, (-1)^n\frac{x^{2n}}{(2n)!}\,,$$

$$V_{2n}(x)\;\;(\text{for } \cosh x) \;=\; 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\, \frac{x^{2n}}{(2n)!}\,.$$

They are so closely related because $\cosh x = \cos(ix)$. In the first case, the absolute value of the remainder term has the upper bound $x^{2n+2}/(2n+2)!$. In the second case, a corresponding bound is $x^{2n+2}\cosh x/(2n+2)!$. If x is restricted to a finite (possibly large) interval covering $x=0$, both remainder terms are bounded by $Ax^{2n+2}/(2n+2)!$ for some constant A and both tend to zero as $n\to\infty$ on such an interval.

Replacing x by $\sqrt{x}$ in the first case and x by $\sqrt{-x}$ in the second gives the same polynomial,

$$T_n(x) \;=\; 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots\, (-1)^n\frac{x^n}{(2n)!}\,,$$

of degree n. This is necessarily the two-sided Taylor polynomial of order n about $x=0$ for the function,

$$G(x) \;=\; \begin{cases} \cos\sqrt{x}\,, & x \ge 0, \\ \cosh\sqrt{-x}\,, & x < 0. \end{cases}$$

The remainder $R_n(x)$ has a bound of the form $A|x|^{n+1}/(2n+2)!$ on any finite interval that covers $x = 0$, and tends to zero as $n \to \infty$ on any such interval. This means that we can identify the polynomial $T_n(x)$ with

$$T_n(x) \;=\; G(0) + G'(0)x + \frac{G''(0)}{2!}x^2 + \frac{G'''(0)}{3!}x^3 + \ldots + \frac{G^{(n)}(0)}{n!}x^n.$$

In particular,

$$G(0) = 0, \quad G'(0) = -\frac{1}{2}, \quad G''(0) = \frac{1}{12}, \quad G'''(0) = -\frac{1}{120}, \quad \ldots,$$

and the nth (two-sided) derivative is

$$G^{(n)}(0) \;=\; (-1)^n \frac{n!}{(2n)!}.$$

(b) The standard Taylor polynomial of order $2n+1$ or $2n+2$ for $\sin x$ about $x = 0$ is

$$U_{2n+1}(x) \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Its remainder term is bounded by $|x|^{2n+3}/(2n+3)!$, which tends to zero as $n \to \infty$ for each fixed $x \in \mathbf{R}$. Dividing by x gives

$$T_{2n}(x) \;=\; 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots + (-1)^n \frac{x^{2n}}{(2n+1)!},$$

which converges to

$$H(x) \;=\; \begin{cases} (\sin x)/x, & x \neq 0 \\ 1, & x = 0, \end{cases}$$

in the limit $n \to \infty$ for each fixed $x \in \mathbf{R}$. So we can identify $T_{2n}(x)$ with the Taylor polynomial,

$$T_{2n}(x) \;=\; H(0) + H'(0)x + \frac{H''(0)}{2!}x^2 + \frac{H'''(0)}{3!}x^3 + \ldots + \frac{H^{(2n)}(0)}{(2n)!}x^{2n}.$$

Because $H(x)$ is even, all the odd-order derivatives of $H(x)$ are zero at $x = 0$. Comparing the two expressions for $T_{2n}(x)$, we get

$$H^{(2n)}(0) \;=\; (-1)^n \frac{(2n)!}{(2n+1)!} \;=\; \frac{(-1)^n}{2n+1}.$$

(c) (Pay attention, there could be a quiz question like this!) The Taylor polynomial of order five or six for $\sin x$ about $x = 0$ is

$$U_5(x) \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Replacing x by x^{10} gives

$$T_{50}(x) \;=\; x^{10} - \frac{x^{30}}{3!} + \frac{x^{50}}{5!}.$$

This is the Taylor polynomial of orders 50, 51, 52, ..., 69 for $f(x) = \sin(x^{10})$ about $x = 0$. So we can read off the derivatives,

$$f^{(10)}(0) \;=\; 10!, \qquad f^{(30)}(0) \;=\; -\frac{30!}{3!}\,, \qquad f^{(50)}(0) \;=\; \frac{50!}{5!}\,.$$

8. (a) Let $f(x) = Ax^2 + Bx + C$. On the interval $[a, b]$ the Mean Value Theorem states that

$$\frac{f(b) - f(a)}{b - a} \;=\; f'(c)$$

for some c such that $a < c < b$. Let us find c explicitly:

$$\begin{aligned} 2Ac + B \;&=\; \frac{Ab^2 + Bb + C - Aa^2 - Ba - C}{b - a} \\ &=\; A(a + b) + B, \end{aligned}$$

which implies $c = (a + b)/2$. So c is unique and is the midpoint of the interval $[a, b]$.

(b) Let $f(x) = Ax^3 + Bx^2 + Cx + D$. The MVT gives

$$\begin{aligned} 3Ac^2 + 2Bc + C \;&=\; \frac{Ab^3 + Bb^2 + Cb + D - Aa^3 - Ba^2 - Ca - D}{b - a} \\ &=\; A(a^2 + ab + b^2) + B(a + b) + C, \\ 3Ac^2 + 2Bc \;&=\; A(a^2 + ab + b^2) + B(a + b). \end{aligned}$$

Substitute the midpoint $c = (a + b)/2$ and subtract the right-hand side from the left-hand side:

$$\begin{aligned} &3A(a + b)^2/4 + B(a + b) - A(a^2 + ab + b^2) - B(a + b) \\ &= (A/4)(3a^2 + 6ab + 3b^2 - 4a^2 - 4ab - 4b^2) \\ &= -A(a - b)^2/4. \end{aligned}$$

Because $A \neq 0$ and $b > a$, this can never vanish. So c can never be the midpoint of $[a, b]$ in the case of a cubic polynomial.

From above, the slope of the chord joining the ends of the cubic arc on $[a, b]$ is $A(a^2 + ab + b^2) + B(a + b) + C$. If this chord is tangent to the cubic curve at $x = b$, then, with $c = b$ above, we get

$$\begin{aligned} 3Ab^2 + 2Bb \;&=\; A(a^2 + ab + b^2) + B(a + b), \\ A(a^2 + ab - 2b^2) + B(a - b) \;&=\; 0, \\ (a - b)\{A(a + 2b) + B\} \;&=\; 0, \\ B \;&=\; -A(a + 2b). \end{aligned}$$

With this constraint on B, the equation above for c becomes

$$\begin{aligned} 3Ac^2 + 2Bc &= A(a^2 + ab + b^2) + B(a + b), \\ 3Ac^2 - 2A(a + 2b)c &= A(a^2 + ab + b^2) - A(a + 2b)(a + b), \\ 3c^2 - 2(a + 2b)c &= -2ab - b^2, \\ (c - b)(3c - 2a - b) &= 0. \end{aligned}$$

The root $c = b$ is out of range because it is an endpoint. The MVT states that c is an interior point. Hence c is unique and takes the value,

$$c = \frac{2a + b}{3}.$$

This is one-third of the way along the interval from a to b.

(c) The Cauchy Mean Value Theorem states that, when $f(x)$ and $g(x)$ are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c)\{g(b) - g(a)\} = g'(c)\{f(b) - f(a)\}.$$

Let $f(x) = \sin x$ and $g(x) = \cos x$. Then

$$\cos c\,(\cos b - \cos a) + \sin c\,(\sin b - \sin a) = 0,$$

$a < c < b$. When $b = a + 2k\pi$, $k \in \mathbf{Z}$, the Cauchy MVT is trivially satisfied by all c in the interval. Exclude this case. Two very useful trig identities here are

$$\begin{aligned} \cos b - \cos a &= -2\sin\Big(\frac{b + a}{2}\Big)\,\sin\Big(\frac{b - a}{2}\Big), \\ \sin b - \sin a &= 2\cos\Big(\frac{b + a}{2}\Big)\,\sin\Big(\frac{b - a}{2}\Big). \end{aligned}$$

The nonzero common factor $\sin\big((b - a)/2\big)$ can be cancelled. Thus

$$\begin{aligned} \sin c\,\cos\Big(\frac{b + a}{2}\Big) - \cos c\,\sin\Big(\frac{b + a}{2}\Big) &= 0, \\ \sin\left(c - \frac{a + b}{2}\right) &= 0. \end{aligned}$$

This equation shows that the midpoint $c = (a + b)/2$ always satisfies the Cauchy MVT for these two functions. It is also satisfied by $c = k\pi + (a + b)/2$, $k \in \mathbf{Z}$, provided $a < c < b$. If $2n\pi < b - a < (2n + 1)\pi$, $n = 0, 1, 2, 3, \ldots$, the number of legal values of c is exactly $2n + 1$.

9. We are given the function,

$$f(x) = \begin{cases} 4x^2 + x, & -1 \le x < 0, \\ 2\sqrt{x}\,, & 0 \le x < 1, \\ (4x^3 - 21x^2 + 36x - 7)/6, & 1 \le x \le 3. \end{cases}$$

on the interval $[-1,3]$. The critical points are the interior points where either $f'(x)=0$ or $f'(x)$ does not exist. All local extrema are included among the critical points. The absolute extrema are included among the local extrema and the endpoints. Since $f(x)$ is defined by three rules, we must check the joins as well as the interiors of each subinterval.

On $[-1,0]$, $f(x)=4x^2+x$ and $f'(x)=8x+1$. There is a critical point at $x=-1/8$, at which $f(-1/8)=-1/16$ and $f''(-1/8)=8$. So a local minimum occurs at $x=-1/8$. At $x=0$, $f(x)$ has a left derivative $f'_-(0)=1$ and so $f(x)$ is increasing as x approaches 0 from the left.

On $[0,1]$, $f(x)=2\sqrt{x}$ and $f'(x)=1/\sqrt{x}$ except for a one-sided vertical tangent at $x=0$. So $x=0$ is a critical point of $f(x)$. It is a corner point through which $f(x)$ is increasing. So $x=0$ is not an extremum of $f(x)$. Since $f(x)$ is strictly increasing on $[0,1]$, there are no critical points in $[0,1)$. The join at $x=1$ needs to be checked. The left derivative there is $f'_-(1)=1$.

On $[1,3]$,

$$\begin{aligned} f(x) &= (4x^3-21x^2+36x-7)/6, \\ f'(x) &= 2x^2-7x+6 = (x-2)(2x-3), \\ f''(x) &= 4x-7. \end{aligned}$$

So critical points occur at $x=3/2$ and $x=2$. The right derivative at $x=1$ is $f'_+(1)=1$, which is the same as the left derivative. So $x=1$ is not a critical point of $f(x)$. At $x=3/2$, $f(3/2)=(1/6)(27/2-189/4+54-7)=53/24$ and $f''(3/2)=-1$. So $f(x)$ has a local maximum at $x=3/2$. At $x=2$, $f(2)=(1/6)(32-84+72-7)=13/6$ and $f''(2)=1$. So $f(x)$ has a local minimum at $x=2$. The endpoint values are $f(-1)=3$ and $f(3)=(108-189+108-7)/6=10/3$. So the absolute minimum occurs at $x=-1/8$, where $f(-1/8)=-1/16$, and the absolute maximum occurs at the right endpoint $x=3$, where $f(3)=10/3$.

The results so far can be summarised as follows:

$x=-1:$ left endpoint, not an extremum;

$x=-1/8:$ critical point, horizontal tangent, local and absolute minimum;

$x=0:$ critical point, corner, right vertical tangent, not an extremum;

$x=1:$ smooth join, not a critical point;

$x=3/2:$ critical point, horizontal tangent, local maximum;

$x=2:$ critical point, horizontal tangent, local minimum;

$x=3:$ right endpoint, absolute maximum.

10. We will describe the level curves (typesetting diagrams is hard work). Level curves are horizontal slices of the given surface that have been dropped down to the xy-plane. They are most informative when a set of level curves are drawn that correspond to equally spaced heights (values of z) in the range of the function. The level curve diagram then forms a contour map of the surface.

(a) $f(x,y) = e^{x^2+y^2}$. The range is $z \geq 1$, so a natural choice for level curve heights is the set of positive integers $z = 1, 2, 3, \ldots$. At height z, the equation of the level curve is

$$x^2 + y^2 = \ln z,$$

which is the equation of a circle, centre $(0,0)$, radius $\sqrt{\ln z}$. When $z = 1, 2, 3, \ldots$, the radii are 0, $\sqrt{\ln 2}$, $\sqrt{\ln 3}$, ... These are a family of concentric circles of increasing radii that are rapidly getting closer together as z increases, which indicates that the surface is getting steeper as the height increases.

(b) $g(x,y) = xy/(x^2+y^2)$. This surface becomes easier to visualize in polar coordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$g(x,y) = \frac{xy}{x^2+y^2} = \frac{r^2\cos\theta\sin\theta}{r^2} = \frac{\sin 2\theta}{2}.$$

The level curves are curves of constant θ, which are, in general, pairs of straight lines through the origin. The origin itself must be cut from every line, because the function is not defined there. (In any case, level curves cannot cross each other, for otherwise the function would fail the vertical line test and be multi-valued.) The range of $g(x,y)$ is $-1/2 \leq z \leq 1/2$. To get a reasonable contour map of this surface, pick $z = -0.50, -0.45, -0.40, \ldots, 0.35, 0.40, 0.45$ and 0.50. The largest z corresponds to 45-degree lines in the first and third quadrants. The smallest z corresponds to 45-degree lines in the second and fourth quadrants. The lines should be drawn closer together near the x and y axes, where the surface is steepest in the transverse direction.

(c) $h(x,y) = (\sqrt{x} + \sqrt{y})^2$. This function has the range $z \geq 0$. So a natural choice of heights is the nonnegative integers $z = 0, 1, 2, 3, \ldots$. The equation of the level curve at height z is

$$\sqrt{x} + \sqrt{y} = \sqrt{z},$$

where the square roots are nonnegative. This curve is a parabolic arc in the first quadrant running from $(z,0)$ on the x-axis to $(0,z)$ on the y-axis, and tangent to both axes. The axis of symmetry of the parabola is the line $y = x$ in the first quadrant. The parabola would continue beyond these contact points if we allowed $\sqrt{x}$ and $\sqrt{y}$ to take both signs. The family of parabolic arcs is equally spaced.

The surface itself is part of a tilted circular cone which touches the three coordinate planes in the first octant. Its vertex is at the origin and its axis is the line $x = y = z$. The line $z = x$, $x \geq 0$, in the xz-plane is a generator of the cone, and similarly for the line $z = y$, $y \geq 0$ in the yz-plane. The line $y = x$ in the xy-plane is a generator of the full cone, but does not belong to the part that is the graph of the function $h(x,y)$. The level curves are parts of the horizontal parabolic conic sections parallel to the latter generator.

Appendix to Question 2

This appendix will carry out the construction of the logarithm and exponential functions to base b and their derivatives from first principles. The starting point will be the original definition of b^x as a power when $b > 0$ and x is an integer, after which it is extended to the rationals and then the reals. The logarithm $\log_b x$, $b > 0$, $b \neq 1$, is defined to be the functional inverse of b^x, and its derivative,

$$\frac{d}{dx} \log_b x = \frac{\log_b e}{x},$$

is found from the limit of the difference quotient. The base of natural logarithms e arises naturally as the limit of $(1 + 1/n)^n$ as $n \to \infty$.

This approach is not the one favoured by most Calculus textbooks, as greater effort is needed to get to the main calculus results involving logarithms and exponentials. However, it is a more fundamental approach, beginning with an intuitive understanding of the notion of a number raised to a power. A few comments about the standard treatment are given at the end.

The definition of b^x for $b > 0$ and $x \in \mathbf{R}$ will be constructed in stages, with a view to preserving the index laws,

$$b^x\, b^y = b^{x+y}, \qquad a^x\, b^x = (ab)^x, \qquad (b^x)^y = b^{xy}.$$

First, if n is a positive integer, then b^n is defined to be the product $b \cdot b \cdot b \cdot \ldots \cdot b$ with n factors. The index laws then require the definitions, $b^0 = 1$ and $b^{-n} = 1/b^n$. Next, if q is a positive integer, the index laws require $b = (b^{1/q})^q$ and so $b^{1/q}$ is the unique positive qth root of b. Then $b^{p/q} = (b^{1/q})^p$ for all $p \in \mathbf{Z}$. This completes the definition of b^r, $b > 0$, for any rational r.

Elementary methods in the differential calculus (not depending on a knowledge of logarithms or exponentials) can produce a proof of

$$\frac{d}{dx} x^r = r\, x^{r-1}, \qquad x > 0,$$

when r is rational. (The result extends to $x = 0$ when $r \geq 1$ and to $x < 0$ when r is an integer or rational number with odd denominator.)

So far, the function $f(b, x) = b^x$ is defined on the domain $b > 0$ and $x \in \mathbf{Q}$, where $\mathbf{Q}$ denotes the rationals. The preceding construction shows that b^x is monotonic, continuous and differentiable in b for each fixed rational x. Also the the identity $(b^{1/q})^q = b$ implies that $b^{1/q} \to 1$ as $q \to \infty$ and, more generally, that $b^k \to 1$ as k tends to zero through the rationals from either side. The index laws then imply that $b^{x+k} \to b^x$ as $k \to 0$ through the rationals, and so b^x is monotonic and continuous as a function of rational x for each fixed $b > 0$.

To complete the construction of b^x for $b > 0$ and unrestricted real x, we invoke continuity. Consider, for example, the sequence,

$$b^{1.4142} = b^{14142/10000}, \quad b^{1.41421}, \quad b^{1.414213}, \quad b^{1.4142135}, \quad b^{1.41421356}, \quad \ldots .$$

Intuitively, we would espect this sequence to tend to a limit as the exponents are taken through increasingly sharper decimal approximations to $\sqrt{2}$. It is then natural to denote the limiting power by the symbol $b^{\sqrt{2}}$. Can this construction define b^x for all real x and is the result continuous and differentiable in both variables?

We need to know when a continuous function $g(x)$ of a rational variable has an extension to a continuous function (which we will also call $g(x)$) of a real variable. If such an extension exists, it is obviously unique. Just to throw a spanner in the works, consider the case $g(x) = 1/(x - \sqrt{2}\,)$ where x runs through the rationals. This function is continuous everywhere on its domain, even though it is unbounded near $x = \sqrt{2}$, which is outside the domain. We see that the unique extension of $g(x)$ to the real domain has an obvious infinite discontinuity at $x = \sqrt{2}$. A problem occurred in this case because $g(x)$ is not uniformly continuous in the rationals near $\sqrt{2}$. In the epsilon-delta definition of limit, the δ depends on both ϵ (as it should) and x (as it usually does). When x is close to $\sqrt{2}$, δ must be chosen correspondingly small so that the interval $(x - \delta,\, x + \delta)$ stays entirely on one side of $\sqrt{2}$. This nonuniformity with respect to x obstructs the real extension of $g(x)$ from including $\sqrt{2}$ in its domain.

The problem observed in the previous paragraph does not occur in the case of $f(b, x) = b^x$, $b > 0$, $x \in \mathbf{R}$. The reason is simply that b^x is monotonic as well as continuous in rational x for each fixed $b > 0$. The monotonicity alone implies that b^{x_n} tends to a limit as $n \to \infty$ whenever x_n tends monotonically to a finite limit, say, $x \in \mathbf{R}$, through the rationals. The continuity implies that the limit b^x is well-defined and independent of the auxiliary sequence of rational powers. In addition, this construction guarantees that b^x is monotonic and continuous as a function of real x for each fixed $b > 0$. Also the previously noted continuity and monotonicity of b^x with respect to b now extends to all fixed real x. (Actually, $f(x, y) = x^y$ is continuous as a function of two real variables in the half-plane $x > 0$, but we will not need this stronger result here.)

Consider the monotonic function b^x for fixed $b > 0$. This function is strictly increasing for $b > 1$ and strictly decreasing for $0 < b < 1$. It is constant for $b = 1$. When $b > 0$ and $b \neq 1$, the function b^x has domain $\mathbf{R}$ and range $\mathbf{R}^+$, the positive reals. If we write $f : \mathbf{R} \to \mathbf{R}^+,\ x \mapsto b^x$, then f so written is bijective. It therefore has an inverse function $f^{-1} : \mathbf{R}^+ \to \mathbf{R},\ x \mapsto \log_b x$, which defines the logarithm $\log_b x$ of $x > 0$ to base b. This function $\log_b x$ is also continuous and monotonic. It is increasing for $b > 1$ and decreasing for $0 < b < 1$. (The logarithm to base 1 does not exist.) The index laws for b^x imply the following logarithm laws ($x,\, y > 0$):

$$\begin{aligned}
\log_b 1 &= 0, \\
\log_b b &= 1, \\
\log_b(xy) &= \log_b x + \log_b y, \\
\log_b(x/y) &= \log_b x - \log_b y, \\
\log_b(x^a) &= a \log_b x, \qquad a \in \mathbf{R}, \\
\log_c x &= \frac{\log_b x}{\log_b c}, \qquad c > 1 \text{ and } 0 < c < 1, \\
\log_c b &= \frac{1}{\log_b c}, \\
\log_{1/b} x &= -\log_b x.
\end{aligned}$$

After that preamble, we are ready to investigate the derivative of $\log_b x$. For $b > 1$ or $0 < b < 1$, the definition of derivative as a limit gives

$$
\begin{aligned}
\frac{d}{dx}\log_b x &= \lim_{h\to 0}\frac{\log_b(x+h)-\log_b x}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\log_b\left(1+\frac{h}{x}\right) \\
&= \lim_{h\to 0}\frac{1}{x}\frac{x}{h}\log_b\left(1+\frac{h}{x}\right) \\
&= \frac{1}{x}\lim_{h\to 0}\log_b\left(1+\frac{h}{x}\right)^{x/h} \\
&= \frac{1}{x}\lim_{k\to\infty}\begin{cases}\log_b\left(1+\dfrac{1}{k}\right)^{k}, & h>0, \\ \log_b\left(1-\dfrac{1}{k}\right)^{-k}, & h<0.\end{cases}
\end{aligned}
$$

The continuity of the logarithm implies the right derivative,

$$\frac{d}{dx}\log_b x = \frac{\log_b e}{x}, \qquad e = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k},$$

and the left derivative,

$$\frac{d}{dx}\log_b x = \frac{\log_b e_1}{x}, \qquad e_1 = \lim_{k\to\infty}\left(1-\frac{1}{k}\right)^{-k},$$

provided these limits exist. Before we go any further, we will show that $e_1 = e$ if the limit defining either e or e_1 exists.

$$\left(1-\frac{1}{k}\right)^{-k} = \left(\frac{k}{k-1}\right)^{k} = \frac{k}{k-1}\left(1+\frac{1}{k-1}\right)^{k-1}.$$

The limit $k\to\infty$ on both sides immediately gives $e_1 = e$, and so the derivative of $\log_b x$ will be two-sided when we show that e exists.

Let us examine the limit defining e more closely. So far k runs through the reals as $k\to\infty$, but we would like to restrict k to just the positive integers $n = 1, 2, 3, \ldots$. If the limit as $k\to\infty$ exists, then, trivially, the limit as $n\to\infty$ is the same. The converse is less obvious. (For example, $\sin\pi n$ tends to a limit while $\sin\pi k$ does not.) It is true that $(1+1/k)^k$ is strictly increasing in the positive real variable k, but the proof is best done after the calculus of logarithms and exponential functions is established, which is not yet the case here in a first-principles construction (see Exercise **4**(e)). Instead, we will show that the limit $k\to\infty$ exists when the limit $n\to\infty$ exists, and that $(1+1/n)^n$ is strictly increasing as n runs through the positive integers.

Suppose, as before, that $n \le k < n+1$. Then,

$$\left(1+\frac{1}{k}\right)^k < \left(1+\frac{1}{k}\right)^{n+1} \le \left(1+\frac{1}{n}\right)^{n+1},$$

$$\left(1+\frac{1}{k}\right)^k \ge \left(1+\frac{1}{k}\right)^{n} > \left(1+\frac{1}{n+1}\right)^{n},$$

$$\frac{n+1}{n+2}\left(1+\frac{1}{n+1}\right)^{n+1} < \left(1+\frac{1}{k}\right)^k < \frac{n+1}{n}\left(1+\frac{1}{n}\right)^{n}.$$

The squeeze lemma shows that, if $(1+1/n)^n$ tends to a limit as $n \to \infty$ through the integers, then $(1+1/k)^k$ tends to the same limit as $k \to \infty$ through the reals. So we may replace the definition of e by

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

The proof that e exists consists in showing that $(1+1/n)^n$ is increasing and bounded above.

$$\begin{aligned}\frac{(1+1/n)^n}{(1+1/(n-1))^{n-1}} &= \left(\frac{n+1}{n}\right)^n\left(\frac{n-1}{n}\right)^{n-1} \\ &= \frac{n}{n-1}\left(1-\frac{1}{n^2}\right)^n.\end{aligned}$$

An application of the Mean Value Theorem to the function x^n, $n = 0, 1, 2, \ldots$, on the interval $[1-a, 1]$ shows that $(1-a)^n \ge 1-na$ for $0 < a < 1$, with equality only when $n = 0, 1$. In particular,

$$\left(1-\frac{1}{n^2}\right)^n > 1-\frac{1}{n} = \frac{n-1}{n},$$

for all integers $n \ge 2$. Hence,

$$\frac{(1+1/n)^n}{(1+1/(n-1))^{n-1}} > \frac{n}{n-1}\,\frac{n-1}{n} = 1,$$

$$\left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{n-1}\right)^{n-1},$$

for all integers $n \ge 2$. This completes the proof that $(1+1/n)^n$ is strictly increasing.

To get an upper bound, use the binomial theorem (for the case of positive integer powers),

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} \\ &= \sum_{k=0}^{n}\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\,\frac{1}{n^k} \\ &< \sum_{k=0}^{n}\frac{n\cdot n\cdot n\cdots n}{k!}\,\frac{1}{n^k}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \frac{n^k}{k!}\,\frac{1}{n^k} \\
&= \sum_{k=0}^{n} \frac{1}{k!} \\
&= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \\
&< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\,.
\end{aligned}
$$

This upper bound is a rapidly convergent infinite series because the factorials $k! = 1 \cdot 2 \cdot 3 \cdots k$ in the denominators grow rapidly. (The first two terms also follow the pattern because $0! = 1$ and $1! = 1$.) It proves that the limit defining e exists, and that

$$
e \le 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\,.
$$

Of course, the series on the right exactly equals e. To complete the proof, we need a lower bound for $(1 + 1/n)^n$. Let m be any integer such that $1 < m < n$. Then

$$
\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} > \sum_{k=0}^{m} \binom{n}{k} \frac{1}{n^k}\,.
$$

Suppose $k \ge 1$. A rather rough lower bound on the binomial coefficients, which will be accurate enough for our purposes, is

$$
\begin{aligned}
\binom{n}{k} \frac{1}{n^k} &= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!\,n^k} \\
&\ge \frac{(n-k+1)^k}{k!\,n^k} \\
&= \frac{1}{k!}\left(1 - \frac{k-1}{n}\right)^k \\
&\ge \frac{1}{k!}\left(1 - \frac{k(k-1)}{n}\right) \\
&> \frac{1}{k!}\left(1 - \frac{m^2}{n}\right).
\end{aligned}
$$

This bound also holds for $k = 0$. Hence, for $1 < m < n$,

$$
\left(1 + \frac{1}{n}\right)^n > \sum_{k=0}^{m} \frac{1}{k!}\left(1 - \frac{m^2}{n}\right).
$$

Of course, this bound is negative and therefore useless if $m > \sqrt{n}$. However, it is just what we need if we let $m = [n^{1/3}]$, where $[x]$ denotes the greatest integer $\le x$. With m so chosen, take the limit as $n \to \infty$ of both the upper and lower bounds on $(1 + 1/n)^n$. The squeeze lemma forces

$$
e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots = 2.71828\,18284\,59045\,23536\ldots\,.
$$

A minor variation of this method produces the everywhere convergent power series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots ,$$

and replacing x by z in this power series defines the complex extension of e^z, preserving the index laws, to the entire complex plane.

To conclude, we have proved from first principles that

$$\frac{d}{dx} \log_b x = \frac{\log_b e}{x} = \frac{1}{(\log_e b)x},$$

for $x > 0$ and $b > 1$ or $0 < b < 1$, where e is the transcendental number evaluated in the preceding paragraph. Differentiating the identity $\log_b(b^x) = x$ gives

$$\frac{d}{dx} b^x = \frac{b^x}{\log_b e} = (\log_e b)b^x.$$

The latter form is valid for all $b > 0$. The derivative formula $(d/dx)x^s = sx^{s-1}$ can be extended from rational to real s according to

$$\frac{d}{dx} x^s = \frac{d}{dx} b^{s \log_b x} = \frac{s \log_b e}{x} \frac{b^{s \log_b x}}{\log_b e} = \frac{s}{x} b^{s \log_b x} = \frac{s}{x} x^s = s\, x^{s-1}.$$

Of course, the particular base $b = e$ is distinguished, because $\log_e e = 1$. Thus,

$$\frac{d}{dx} \log_e x = \frac{1}{x}, \qquad \frac{d}{dx} e^x = e^x.$$

The function $\log_e x$ is also denoted $\ln x$ (a notation apparently first used in 1893), which stands for "logarithmus naturalis", a name coined by Nicholas Mercator in a treatise published in 1668. The most compelling reasons to consider e to be the natural base of logarithms and exponentials comes from the calculus and Taylor series and later applications. But Mercator already saw the advantages of this base well before the calculus was invented. Mathematicians use the notation $\log x$ (without a base specified) to mean $\log_e x$ or $\ln x$. However, before the widespread use of pocket calculators in the 1970s, $\log x$ could also have meant the common logarithm $\log_{10} x$, which was used in numerical computations and whose values could be looked up in tables. Pocket calculators usually use "LN" to denote $\log_e$ and "LOG" to denote $\log_{10}$. Common logarithms and binary logarithms (base 2) continue to be important in number theory and computer science.

Concluding remarks. This "first principles" construction of $\log_b x$ and its derivative turned out to be quite long, but not unexpectedly so. Many of the familiar results of elementary calculus stand on top of a lot of pre-calculus infrastructure which one often takes for granted. A corresponding first-principles derivation of, say, $e^{i\pi} = -1$ starting from nothing would take about ten pages. It would be quick if π was defined as the smallest positive zero of the function $\sin x$ defined by its Taylor series. But, of course, neither π nor $\sin x$ was originally defined this way, π being the ratio of the circumference of a circle to its diameter and $\sin\theta$ being the ratio of the opposite to the hypotenuse in a right-angled triangle with angle θ (with the angles measured in degrees initially).

The usual starting point in Calculus textbooks for the construction of the functions $\log_b x$ and b^x is the function $\ln x$ defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

Students would certainly be justified in questioning this rather arcane starting point. After all, an integral is a complicated limit with many moving parts. But this approach gets to the main results about logarithms and exponentials much more quickly than the more fundamental starting point in this appendix. By inspection, $\ln x$ so defined is continuously differentiable, having derivative $1/x$, is strictly increasing, and is bijective as a function from $\mathbf{R}^+$ to $\mathbf{R}$. Some elementary operations on the integral show that $\ln x$ obeys the logarithm laws, and is in fact a logarithm to base e, where e is defined to be the unique zero of the function $\ln x - 1$. The inverse function is denoted $\exp x$, which is also continuously differentiable and increasing, and e is now identified with $\exp 1$. After that $\exp x$ is shown to obey the index laws and is identified with the power e^x. In addition $(d/dx)e^x = e^x$ follows from $(d/dx)\ln x = 1/x$.

Finally, logarithms and exponentials to any base $b > 0$ ($b \neq 1$ in the case of the logarithm) are defined by

$$\log_b x = \frac{\ln x}{\ln b}, \qquad b^x = \exp(x \ln b).$$

These also obey all the logarithm and index laws, and, of course, agree with $\log_b x$ and b^x as constructed in this appendix. However, taking their derivatives is now a very simple exercise. Finally, to evaluate $\lim\limits_{k\to\infty} (1 + 1/k)^k$, replace k by $1/x$ and define

$$E = \lim_{x\to 0}(1 + x)^{1/x}.$$

Then, by l'Hôpital's rule, the logarithm laws, and the continuity of $\ln x$,

$$\ln E = \lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1/(1+x)}{1} = 1,$$

which implies $E = e$. Since the limit is two-sided, we have proved

$$\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e, \qquad \lim_{x\to\infty}\left(1 - \frac{1}{x}\right)^{-x} = e,$$

in agreement with the results found above. The limit through the integers is a corollary of the limit through the reals (the converse of what we needed above). More generally, the same method gives

$$\lim_{x\to\infty}\left(1 + \frac{a}{x}\right)^{bx} = e^{ab}, \qquad a,\, b \in \mathbf{R},$$

which can be deduced directly from the cases $a = b = \pm 1$.