

THE UNIVERSITY OF SYDNEY
MATH1902 LINEAR ALGEBRA (ADVANCED)

Semester 1

Solutions to Assignment 1

2009

1. Consider the following list of vector “identities”, some of which are true, some false, and others meaningless. Identify which is which. If an identity is meaningless, explain why it is so. If an identity is false, give specific values of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ which demonstrate its falsity. If an identity is true, then prove it, using either the algebraic or geometric definition of the dot and/or cross products.

- (a) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$
- (b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
- (c) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
- (d) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$
- (e) $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ together imply $\mathbf{b} = \mathbf{c}$
- (f) $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ together imply $\mathbf{b} = \mathbf{c}$
- (g) $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ together imply $\mathbf{b} = \mathbf{c}$
- (h) $|\mathbf{a} \times \mathbf{b}|^2 + |\mathbf{a} \cdot \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$

Solution.

- (a) This identity is meaningless. Since \mathbf{a} is a vector and $\mathbf{b} \cdot \mathbf{c}$ is a scalar, the dot product $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$ is undefined. (Of course $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ is undefined also.)
- (b) This identity is false. For example, take $\mathbf{a} = \mathbf{b} = \mathbf{i}$ and $\mathbf{c} = \mathbf{j}$. We have

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}, \quad \text{while} \quad (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}.$$

- (c) This identity is true. Write $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. We have

$$\mathbf{b} \times \mathbf{c} = (b_2c_3 - b_3c_2)\mathbf{i} + (b_3c_1 - b_1c_3)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k},$$

so that

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1). \end{aligned}$$

A similar calculation shows that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (a_2b_3c_1 + a_3b_1c_2 + a_1b_2c_3) - (a_3b_2c_1 + a_1b_3c_2 + a_2b_1c_3),$$

which is easily seen to be equal to our expression above for $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

- (d) This identity is meaningless, for similar reasons to that from part (a).

(e) Note that with $\mathbf{a} \neq \mathbf{0}$, we have

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \iff \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0 \iff \mathbf{b} - \mathbf{c} = \mathbf{0} \text{ or } \mathbf{a} \perp \mathbf{b} - \mathbf{c}.$$

Thus, choosing $\mathbf{a}, \mathbf{b}, \mathbf{c}$ so that $\mathbf{a} \perp \mathbf{b} - \mathbf{c} \neq \mathbf{0}$ will give us a counterexample. For example, take $\mathbf{a} = \mathbf{i}, \mathbf{b} = \mathbf{j}, \mathbf{c} = \mathbf{0}$.

(f) This identity is false. By a similar line of reasoning to the previous part, if $\mathbf{a} \neq \mathbf{0}$, then

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \iff \mathbf{b} - \mathbf{c} = \mathbf{0} \text{ or } \mathbf{a} \parallel \mathbf{b} - \mathbf{c}.$$

So we may choose any $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{a} \parallel \mathbf{b} - \mathbf{c} \neq \mathbf{0}$, for example $\mathbf{a} = \mathbf{b} = \mathbf{i}$ and $\mathbf{c} = \mathbf{0}$.

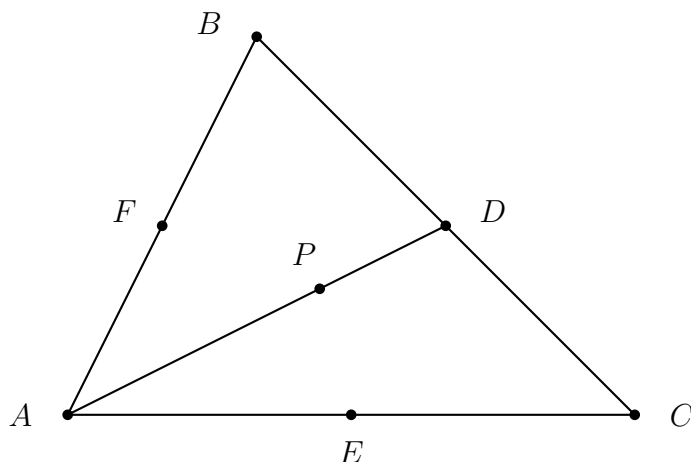
(g) This identity is true. To see this, suppose that $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ all hold. By the argument in part (e) above, we conclude that either $\mathbf{b} = \mathbf{c}$ or $\mathbf{a} \perp \mathbf{b} - \mathbf{c}$. By part (f), we have either $\mathbf{b} = \mathbf{c}$ or $\mathbf{a} \parallel \mathbf{b} - \mathbf{c}$. Since two non-zero vectors cannot be simultaneously perpendicular and parallel, we conclude that $\mathbf{b} = \mathbf{c}$.

(h) This identity is true. Let θ denote the (non-reflex) angle between \mathbf{a} and \mathbf{b} (when placed tail-to-tail). Then $\mathbf{a} \cdot \mathbf{b} = ab \cos \theta$ and $|\mathbf{a} \times \mathbf{b}| = ab \sin \theta$. (Here we have written $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$.) Thus,

$$|\mathbf{a} \times \mathbf{b}|^2 + |\mathbf{a} \cdot \mathbf{b}|^2 = a^2 b^2 \cos^2 \theta + a^2 b^2 \sin^2 \theta = a^2 b^2 (\cos^2 \theta + \sin^2 \theta) = a^2 b^2 = |\mathbf{a}|^2 |\mathbf{b}|^2.$$

2. A *median* of a triangle is a line that connects a vertex to the midpoint of its opposite side. Show that the medians of any triangle intersect at a point. Also show that this point divides each median in the ratio 2 : 1 (measured from the vertex to the midpoint of the opposite side).

Solution. (Several other solutions exist.) Label the vertices of the triangle A, B, C and let D, E, F denote the midpoints of the sides opposite A, B, C respectively. Let P be the point that divides the median AD in the ratio 2 : 1.



We first show that $\overrightarrow{BP} = \frac{2}{3}\overrightarrow{BE}$. Indeed, we calculate

$$\begin{aligned}
 \overrightarrow{BP} &= \overrightarrow{BA} + \overrightarrow{AP} \\
 &= \overrightarrow{BA} + \frac{2}{3}\overrightarrow{AD} && \text{since } AP : PD = 2 : 1 \\
 &= \overrightarrow{BA} + \frac{2}{3} \cdot \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC}) && \text{since } D \text{ is the midpoint of } BC \\
 &= \overrightarrow{BA} - \frac{1}{3}\overrightarrow{BA} + \frac{1}{3}\overrightarrow{AC} && \text{since } \overrightarrow{AB} = -\overrightarrow{BA} \\
 &= \frac{2}{3}\overrightarrow{BA} + \frac{2}{3}\overrightarrow{AE} && \text{since } \overrightarrow{AC} = 2\overrightarrow{AE} \\
 &= \frac{2}{3}(\overrightarrow{BA} + \overrightarrow{AE}) \\
 &= \frac{2}{3}\overrightarrow{BE}.
 \end{aligned}$$

This has two implications:

- (i) that \overrightarrow{BP} is parallel to \overrightarrow{BE} , which means that P lies on the median BE , and
- (ii) that P divides BE in the ratio $2 : 1$.

A similar calculation shows that $\overrightarrow{CP} = \frac{2}{3}\overrightarrow{CF}$ which shows that P also lies on the median CF , dividing it in the ratio $2 : 1$. This completes the proof.

3. In this question we will derive a formula for the (shortest) distance between two non-parallel, non-intersecting lines.

- (a) Suppose that \mathcal{P}_1 and \mathcal{P}_2 are parallel planes. Let \mathbf{w} be a vector perpendicular to both planes, and suppose that A and B are points on \mathcal{P}_1 and \mathcal{P}_2 respectively. Show that the distance between the two planes is given by

$$\frac{|\overrightarrow{AB} \cdot \mathbf{w}|}{|\mathbf{w}|}.$$

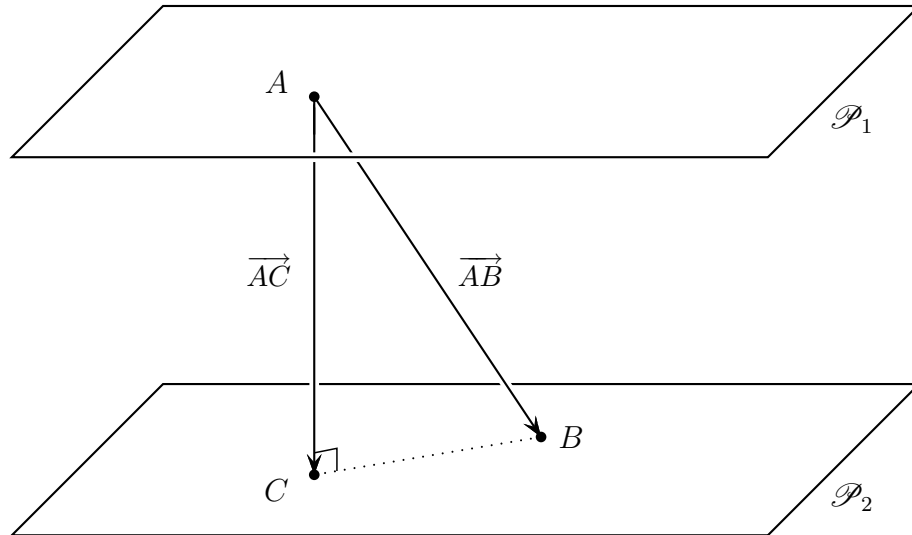
(The distance between the planes is defined to be the minimum distance between a point from \mathcal{P}_1 and a point from \mathcal{P}_2 .)

- (b) Let \mathcal{L}_1 and \mathcal{L}_2 be non-parallel, non-intersecting lines in space. Suppose that A and B are points on \mathcal{L}_1 and \mathcal{L}_2 (respectively) and that \mathbf{u} and \mathbf{v} are vectors parallel to \mathcal{L}_1 and \mathcal{L}_2 (respectively).
 - (i) Show that there exist points P and Q on \mathcal{L}_1 and \mathcal{L}_2 (respectively) such that \overrightarrow{PQ} is perpendicular to both \mathcal{L}_1 and \mathcal{L}_2 .
 - (ii) Show that the distance between \mathcal{L}_1 and \mathcal{L}_2 is equal to

$$\frac{|\overrightarrow{AB} \cdot (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|}.$$

Solution.

(a) Let C be the point on \mathcal{P}_2 that is closest to A , as in the following diagram:



Obviously C is the (unique) point on \mathcal{P}_2 for which \overrightarrow{AC} is perpendicular to both planes, and the distance d between the two planes is equal to the length of \overrightarrow{AC} . Note that, regardless of whether \mathbf{w} points from \mathcal{P}_1 to \mathcal{P}_2 or vice versa, the vector \overrightarrow{AC} is the projection of \overrightarrow{AB} in the direction of \mathbf{w} , so we have

$$\overrightarrow{AC} = \frac{\overrightarrow{AB} \cdot \mathbf{w}}{w} \hat{\mathbf{w}}.$$

We then have

$$d = |\overrightarrow{AC}| = \left| \frac{\overrightarrow{AB} \cdot \mathbf{w}}{w} \right| \cdot |\hat{\mathbf{w}}| = \frac{|\overrightarrow{AB} \cdot \mathbf{w}|}{w},$$

completing the proof.

(b) Throughout this question if we use a capital letter, say R , to denote a point, we will write the corresponding bold-faced lower-case letter to denote its position vector (so $\mathbf{r} = \overrightarrow{OR}$ in this case).

(i) Note that any point P on \mathcal{L}_1 satisfies

$$\mathbf{p} = \mathbf{a} + s\mathbf{u}$$

for some scalar s . Likewise, if Q is a point on \mathcal{L}_2 , then

$$\mathbf{q} = \mathbf{b} + t\mathbf{v}$$

for some t . We must show that there exist scalars s and t such that the vector

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = (\mathbf{b} - \mathbf{a}) + t\mathbf{v} - s\mathbf{u}$$

is perpendicular to both \mathcal{L}_1 and \mathcal{L}_2 . Now,

$$\begin{aligned}\overrightarrow{PQ} \perp \mathcal{L}_1 &\iff \overrightarrow{PQ} \perp \mathbf{u} \\ &\iff \overrightarrow{PQ} \cdot \mathbf{u} = 0 \\ &\iff (\mathbf{b} - \mathbf{a}) \cdot \mathbf{u} + t(\mathbf{v} \cdot \mathbf{u}) - s(\mathbf{u} \cdot \mathbf{u}) = 0 \\ &\iff (\mathbf{u} \cdot \mathbf{u})s - (\mathbf{u} \cdot \mathbf{v})t = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{u}.\end{aligned}$$

A similar calculation shows that

$$\overrightarrow{PQ} \perp \mathcal{L}_2 \iff (\mathbf{u} \cdot \mathbf{v})s - (\mathbf{v} \cdot \mathbf{v})t = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{v}.$$

For simplicity, let us write

$$\alpha = \mathbf{u} \cdot \mathbf{u} = u^2, \quad \beta = \mathbf{u} \cdot \mathbf{v}, \quad \gamma = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{u}, \quad \delta = \mathbf{v} \cdot \mathbf{v} = v^2, \quad \varepsilon = (\mathbf{b} - \mathbf{a}) \cdot \mathbf{v}.$$

The previous calculations show that the vector \overrightarrow{PQ} is perpendicular to both lines if and only if the system of (linear) equations

$$\alpha s - \beta t = \gamma \tag{1}$$

$$\beta s - \delta t = \varepsilon \tag{2}$$

has a solution for s and t . Thus, the proof will be complete if we can show that a solution does exist. Now $\alpha = u^2 \neq 0$ (since $\mathbf{u} \neq \mathbf{0}$, being the direction vector of the line \mathcal{L}_1), so we may multiply Equation (1) by $\frac{\beta}{\alpha}$ to obtain:

$$\beta s - \frac{\beta^2}{\alpha} t = \frac{\beta\gamma}{\alpha}. \tag{3}$$

Subtracting Equation (2) from Equation (3) gives

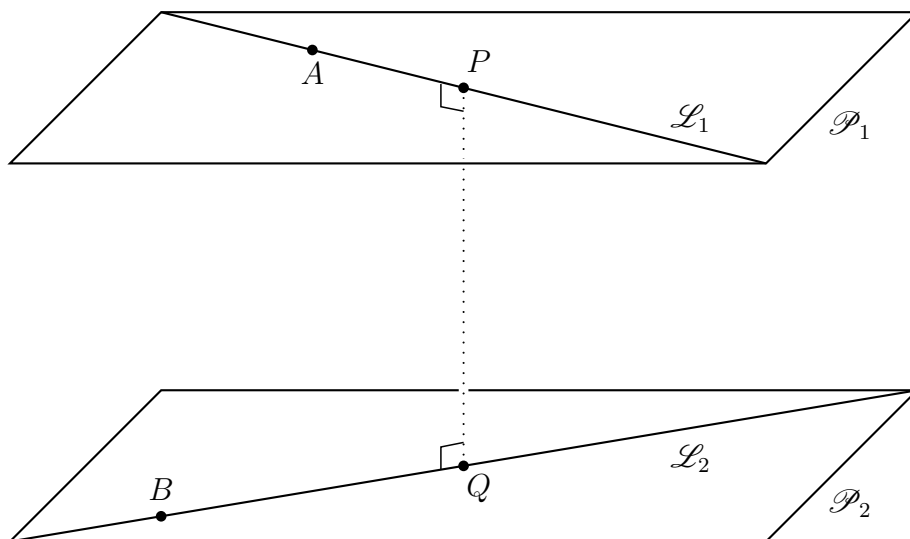
$$\left(\delta - \frac{\beta^2}{\alpha}\right) t = \frac{\beta\gamma}{\alpha} - \varepsilon. \tag{4}$$

This equation has a (unique) solution for t since, writing θ for the angle between \mathbf{u} and \mathbf{v} (when placed tail-to-tail), we have

$$\begin{aligned}\delta - \frac{\beta^2}{\alpha} &= v^2 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{u^2} \\ &= v^2 - \frac{u^2 v^2 \cos^2 \theta}{u^2} \\ &= v^2(1 - \cos^2 \theta) \\ &= v^2 \sin^2 \theta\end{aligned}$$

which is non-zero since $v \neq 0$ (as $\mathbf{v} \neq \mathbf{0}$) and $\theta \neq 0^\circ, 180^\circ$ (as $\mathcal{L}_1 \not\parallel \mathcal{L}_2$ and hence $\mathbf{u} \not\parallel \mathbf{v}$). Equation (1) may now be used to find the corresponding value of s . This completes the proof.

- (ii) Let \mathcal{P}_1 be the plane through A , perpendicular to \overrightarrow{PQ} (where P and Q are the points from the previous part). Similarly, let \mathcal{P}_2 be the plane through B , perpendicular to \overrightarrow{PQ} . Note that \mathcal{P}_1 contains \mathcal{L}_1 and that \mathcal{P}_2 contains \mathcal{L}_2 , since \overrightarrow{PQ} is perpendicular to both lines. See the diagram below.



Now \overrightarrow{PQ} being perpendicular to \mathcal{P}_1 and \mathcal{P}_2 implies that the two planes are parallel, and that the distance d between them is equal to $|\overrightarrow{PQ}|$. But since P is on \mathcal{L}_1 and Q is on \mathcal{L}_2 , it also follows that the (shortest) distance between \mathcal{L}_1 and \mathcal{L}_2 is equal to d . Now by part (a), we have

$$d = \frac{|\overrightarrow{AB} \cdot \mathbf{w}|}{|\mathbf{w}|}$$

where \mathbf{w} is any vector perpendicular to \mathcal{P}_1 and \mathcal{P}_2 . Now $\mathbf{u} \times \mathbf{v}$ is perpendicular to both \mathbf{u} and \mathbf{v} and, as such, is parallel to \overrightarrow{PQ} and therefore perpendicular to both planes. So we may take $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, and we conclude that the distance between \mathcal{L}_1 and \mathcal{L}_2 is equal to

$$\frac{|\overrightarrow{AB} \cdot (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|}$$

as required.

[Note that we *could* have taken $\mathbf{w} = \overrightarrow{PQ}$ above, but the resulting expression, $\frac{|\overrightarrow{AB} \cdot \overrightarrow{PQ}|}{|\overrightarrow{PQ}|}$ (which simplifies to $|\overrightarrow{PQ}|$), is not as useful in practice since it requires us to first calculate the points P and Q explicitly.]