The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1	Assignment 2 Solutions	2009

1. The matrix C is given by

$$C = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & a & b & 3 \\ -2 & -4 & a & -6 \\ a & 4 & -a & 6 \end{bmatrix},$$

where a and b are constants.

(*i*) Calculate the determinant of C.

(*ii*) Find the values of a and b for which the matrix C is not invertible. Calculate the rank of C for each of these values.

(*iii*) Give the general solution of the system of equations $C\mathbf{x} = \mathbf{0}$ for the values of a and b for which $\operatorname{rank} C = 2$.

Solution.

(*i*) Apply the elementary row operations $R_2 := R_2 - R_1$, $R_3 := R_3 + 2R_1$ and $R_4 := R_4 - aR_1$ to the matrix C. Since these operations do not change the determinant, we obtain

$$\det C = \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & a-2 & b+1 & 0 \\ 0 & 0 & a-2 & 0 \\ 0 & 4-2a & 0 & 6-3a \end{vmatrix}.$$

Now use the first column expansion to get

$$\det C = \begin{vmatrix} a-2 & b+1 & 0 \\ 0 & a-2 & 0 \\ 4-2a & 0 & 6-3a \end{vmatrix} = (a-2)^2(6-3a) = -3(a-2)^3.$$

(*ii*) Hence, C is not invertible if and only if $a = 2$ and b is arbitrary. If $a = 2$ then applying the elementary row operations to the matrix C as in part (i), we get the matrix

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & b+1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, if $b = -1$ then $\operatorname{rank} C = 1$ and if $b \neq -1$ then $\operatorname{rank} C = 2$.

(*iii*) By the previous part, $b \neq -1$ and the equations read

$$x_1 + 2x_2 - x_3 + 3x_4 = 0$$
$$(b+1)x_3 = 0.$$

Hence, the free variables are x_2 and x_4 and the general solution is

$$x_1 = -2t - 3s, \quad x_2 = t, \quad x_3 = 0, \quad x_4 = s,$$

where t and s are arbitrary real numbers.

2. Let A and B be square matrices of the same size.

(i) Prove that the relation $(A+B)^2 = A^2 + 2AB + B^2$ implies $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.

(ii) Suppose A is invertible. Decide whether the following implications hold:

(a) $(AB)^2 = A^2B^2$ implies $(AB)^3 = A^3B^3$.

(b) $(AB)^2 = A^2B^2$ implies $A(B^2A)^2B^2 = B^3A(BA)^2B$.

(c) $(AB)^2 = A^2B^2$ implies $(BA)^2 = A^2B^2$.

For each of (a), (b) and (c), provide a proof or a counterexample.

Solution.

(i) We have $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$. Hence, the assumption gives $AB = BA$. Then $(A+B)^3 = (A+B)^2(A+B) = (A^2 + 2AB + B^2)(A+B) = A^3 + 2ABA + B^2A + A^2B + 2AB^2 + B^3$. Since $AB = BA$, this coincides with $A^3 + 3A^2B + 3AB^2 + B^3$.

(ii) (a) This implication holds. We have $ABAB = A^2B^2$. Multiplying by A^{-1} from the left we get $BAB = AB^2$. This means that B commutes with AB. Hence, $B^kAB = AB^{k+1}$ for any positive power k. Therefore

$$(AB)^3 = ABABAB = A^2B^2AB = A^3B^3,$$

as required.

(b) This implication holds too. Using the relation $B^kAB = AB^{k+1}$ repeatedly, we get

$$A(B^2A)^2B^2 = AB^2AB^2AB^2 = A^2B^4AB^2 = A^3B^6$$

and

$$B^3A(BA)^2B = B^3ABABAB = AB^4ABAB = A^2B^5AB = A^3B^6,$$

as required.

(c) This statement is false. Indeed, since $BAB = AB^2$, the relation $(BA)^2 = A^2B^2$ is equivalent to $AB^2A = A^2B^2$, and this is also equivalent to $B^2A = AB^2$ because A is invertible. Thus, in order to give a counterexample we need to produce an invertible matrix A and a matrix B such that B commutes with AB, but A and B^2 do not commute. (Note that B has to be not invertible; for invertible matrices B the implication (c) does hold). Take

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then A is invertible and we have $AB = B$ and $B^2 = B$. Therefore B commutes with AB while $AB^2 = AB = B$ and

$$B^2A = BA = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

so that $AB^2 \neq B^2A$.