THE UNIVERSITY OF SYDNEY

MATH1902 Linear Algebra (advanced)

Exercises for Week 3 (beginning March 16) Semester 1 2009

Preliminary Reading:

Chapter 3 of the Vectors book.

Objectives:

By the end of Week 3, to achieve at least a pass level, you should

3A: be able to find the length and direction cosines of a vector in three dimensions

3B: be able to calculate the scalar product of two vectors

3C: be able to calculate the projection of a vector $\mathbf{v}$ in the direction of a vector $\mathbf{u}$

3D: be able to calculate the vector product of two vectors.

To achieve higher than a pass level you should

3F: be able to use scalar and vector products to derive theorems in geometry.

3G: be able to carry out algebraic manipulations involving both scalar and vector products.

Preparatory questions. (Answers are on the next page.)

1. The vertices of a triangle are the points $A\,(2,-1,-3)$, $B\,(4,2,3)$ and $C\,(6,3,4)$. Find

(i) the Cartesian coordinates of the vectors $\vec{AB}$ and $\vec{AC}$,

(ii) the lengths of the vectors found in (i),

(iii) the direction cosines of the vectors found in (i).

2. Given the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, calculate

(i) $\mathbf{u} \cdot \mathbf{v}$;

(ii) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$;

(iii) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$;

(iv) the (scalar) component of $\mathbf{v}$ in the direction of $\mathbf{u}$;

(v) the (vector) projection of $\mathbf{v}$ in the direction of $\mathbf{u}$.

3. Given the vectors $\mathbf{u} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, find

(i) the lengths of $\mathbf{u}$ and $\mathbf{v}$;

(ii) $\mathbf{u} \times \mathbf{v}$;

(iii) the sine of the angle θ between $\mathbf{u}$ and $\mathbf{v}$;

(iv) $(3\mathbf{u} - 2\mathbf{v}) \times (\mathbf{u} + 5\mathbf{v})$.

Self-assessment checklist

Tick the box or boxes and seek help from your tutor, if required.

☐ I was unable to complete the Preparatory Questions.

☐ I completed the Preparatory Questions:

☐ with ease. ☐ with some effort. ☐ with difficulty.

Practice questions

4. Show that $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ are perpendicular. Find a vector of unit length that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

5. (*i*) Show that, for all values of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{h}$,

$$(\mathbf{a}-\mathbf{b})\cdot(\mathbf{h}-\mathbf{c}) + (\mathbf{b}-\mathbf{c})\cdot(\mathbf{h}-\mathbf{a}) + (\mathbf{c}-\mathbf{a})\cdot(\mathbf{h}-\mathbf{b}) = 0.$$

(*ii*) Use this identity to show that the altitudes of any triangle ABC meet in a point H. (An *altitude* of the triangle ABC is a line from one of the vertices (say A) to a point on the opposite side that is perpendicular to that side (i.e. perpendicular to BC). The point H where all three altitudes meet is called the *orthocentre* of the triangle).
[Hint: If two terms of the LHS of the above identity are zero, the third term will also be zero.]

(*iii*) Show that, for all values of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{k}$,

$$(\mathbf{a}-\mathbf{b})\cdot\left(\mathbf{k}-\frac{\mathbf{a}+\mathbf{b}}{2}\right) + (\mathbf{b}-\mathbf{c})\cdot\left(\mathbf{k}-\frac{\mathbf{b}+\mathbf{c}}{2}\right) + (\mathbf{c}-\mathbf{a})\cdot\left(\mathbf{k}-\frac{\mathbf{c}+\mathbf{a}}{2}\right) = 0.$$

(*iv*) Use this identity to show that the perpendicular bisectors of the sides of any triangle ABC meet in a point K (the circumcentre of the triangle).

6. Consider the vector triple product $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$.

(*i*) For any vectors $\mathbf{U}$ and $\mathbf{V}$, $\mathbf{U}\times\mathbf{V}$ is perpendicular to both $\mathbf{U}$ and $\mathbf{V}$. Use this fact to show that $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ lies in the plane of the vectors $\mathbf{A}$ and $\mathbf{B}$, so that

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \lambda\mathbf{A} + \mu\mathbf{B},$$

for some scalars λ and μ.

(*ii*) To find λ and μ, take $\mathbf{i}$ to be the unit vector in the direction of $\mathbf{A}$, and take $\mathbf{j}$ as the unit vector perpendicular to $\mathbf{i}$ in the plane of $\mathbf{A}$ and $\mathbf{B}$. Indicate why it is true that

$$\mathbf{A} = a\mathbf{i}$$
$$\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}.$$

(*iii*) By writing $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, where $\mathbf{k} = \mathbf{i}\times\mathbf{j}$, compute $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ directly and find λ and μ. Hence prove

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}.$$

Answers to Preparatory Questions

1. (*i*) $(2,3,6)$ and $(4,4,7)$. (*ii*) 7 and 9.
(*iii*) For $\overrightarrow{AB}$: 2/7, 3/7 and 6/7. For $\overrightarrow{AC}$: 4/9, 4/9 and 7/9.

2. (i) -9 (ii) $-3/\sqrt{14}$ (iii) -5 (iv) -3 (v) $-\mathbf{i}+2\mathbf{j}-2\mathbf{k}$.

3. (i) $|\mathbf{u}| = |\mathbf{v}| = \sqrt{14}$.

(ii) $-4\mathbf{i}-8\mathbf{j}-10\mathbf{k}$.

(iii) $\sin\theta = \dfrac{6\sqrt{5}}{\sqrt{14}\sqrt{14}} = \dfrac{3\sqrt{5}}{7}$.

(iv) This simplifies to $17(\mathbf{u}\times\mathbf{v})$, which equals $17(-4\mathbf{i}-8\mathbf{j}-10\mathbf{k})$.

Self-assessment checklist:

Think about the work you have completed and how it relates to the objectives on the first page. This is aimed at helping you focus on how well you are going and on the areas in which you may need to do further practice or seek assistance.

In the following table, each row corresponds to one of the objectives listed on the first page. Tick the box corresponding to the level of understanding you believe you have achieved.

My understanding is:	Nil	Small	Good	Very Good	Complete
Objective 3A	☐	☐	☐	☐	☐
Objective 3B	☐	☐	☐	☐	☐
Objective 3C	☐	☐	☐	☐	☐
Objective 3D	☐	☐	☐	☐	☐
Objective 3E	☐	☐	☐	☐	☐
Objective 3F	☐	☐	☐	☐	☐

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 3.