THE UNIVERSITY OF SYDNEY
MATH1902 Linear Algebra (advanced)

Exercises for Week 8 (beginning April 27) Semester 1 2009

Preliminary Reading:

Chapter 2 of the Linear Algebra book.

Objectives:

By the end of Week 8, to achieve at least a pass level, you should be able to

8A: find the inverse of a 2×2 matrix by direct calculation,

8B: manipulate expressions given in $\sum$-notation,

8C: use row operations to compute the inverse of a matrix.

To achieve higher than a pass level you should be able to

8D: relate elementary row operations and elementary matrices,

8E: employ elementary matrices to prove facts about matrices in general,

8F: solve simple problems involving "abstract" matrices.

Preparatory questions. (Answers are on the next page.)

1. Find the inverse of the matrix $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$.

2. Simplify the expression $\displaystyle\sum_{i=1}^{3}\sum_{j=1}^{2}\sum_{k=1}^{2} a_{ij}\delta_{jk}b_{ki}$, where δ_{jk} is the Kronecker delta.

3. Find the inverse of $\begin{bmatrix} 1 & -2 & -1 \\ -3 & 5 & 1 \\ 10 & -12 & 8 \end{bmatrix}$.

4. Find the inverse of $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & -3 & -7 \end{bmatrix}$.

Self-assessment checklist

Tick the box or boxes and seek help from your tutor, if required.

☐ I was unable to complete the Preparatory Questions.

☐ I completed the Preparatory Questions:

☐ with ease. ☐ with some effort. ☐ with difficulty.

Practice questions

5. For each of the following matrices, determine whether it is invertible, and find the inverse if there is one.

(*i*) $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ (*ii*) $\begin{bmatrix} 0 & 1 & 2 \\ -3 & 0 & 3 \\ -2 & -1 & 0 \end{bmatrix}$

6. Use the previous question to solve the following system of linear equations:

$$\begin{aligned} 2x - z &= 2 \\ x + 3z &= 1 \\ x + 2y + z &= -1 \end{aligned}$$

7. Use elementary row operations to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 6 & 3 & 0 \end{bmatrix},$$

and hence express A as a product of elementary matrices. (From 1996 exam.)

8. Let X be a square matrix and suppose that for some real numbers $t_0, t_1, \ldots, t_n$,

$$t_n X^n + t_{n-1} X^{n-1} + \cdots + t_1 X + t_0 I = \mathbf{0}.$$

Show that if $t_0 \neq 0$ then X is invertible. (Hint: move the term $t_0 I$ to the right hand side and factorise the left hand side.)

9. Let J be the $n \times n$ matrix each of whose entries is 1.

(*i*) Show that $J^2 = nJ$.

(*ii*) Show that if $n > 1$ then $(I - J)^{-1} = I - \dfrac{1}{n-1} J$.

10. Let A be an $n \times n$ matrix whose (i, j)-th entry is a_{ij}. We say that A is *upper triangular* if $a_{ij} = 0$ whenever $i > j$. Show that the product of two $n \times n$ upper triangular matrices is upper triangular. (Do the case $n = 3$ before attempting the general case.)

Answers to Preparatory Questions

1. $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$.

2. $a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22} + a_{31}b_{13} + a_{32}b_{23}$.

3. The inverse is $\begin{bmatrix} -26 & -14 & -\frac{3}{2} \\ -17 & -9 & -1 \\ 7 & 4 & \frac{1}{2} \end{bmatrix}$, as shown by the following sequence of elementary row operations:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 & 1 & 0 \\ 10 & -12 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3:=R_3-10R_1]{R_2:=R_2+3R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 3 & 1 & 0 \\ 0 & 8 & 18 & -10 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2:=(-1)R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & -1 & 0 \\ 0 & 8 & 18 & -10 & 0 & 1 \end{array}\right] \xrightarrow{R_3:=R_3-8R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & -1 & 0 \\ 0 & 0 & 2 & 14 & 8 & 1 \end{array}\right]$$

$$\xrightarrow{R_3:=(1/2)R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & -1 & 0 \\ 0 & 0 & 1 & 7 & 4 & \frac{1}{2} \end{array}\right] \xrightarrow[R_1:=R_1+R_3]{R_2:=R_2-2R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 8 & 4 & \frac{1}{2} \\ 0 & 1 & 0 & -17 & -9 & -1 \\ 0 & 0 & 1 & 7 & 4 & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_1:=R_1+2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -26 & -14 & -\frac{3}{2} \\ 0 & 1 & 0 & -17 & -9 & -1 \\ 0 & 0 & 1 & 7 & 4 & \frac{1}{2} \end{array}\right]$$

4. Note that the 2×2 blocks in the inverse are the inverses of the 2×2 blocks in the given matrix and so the inverse is $\begin{bmatrix} 3 & -2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 3 & 2 \end{bmatrix}$.

Self-assessment checklist:

Think about the work you have completed and how it relates to the objectives on the first page. This is aimed at helping you focus on how well you are going and on the areas in which you may need to do further practice or seek assistance.

In the following table, each row corresponds to one of the objectives listed on the first page. Tick the box corresponding to the level of understanding you believe you have achieved.

My understanding is:	Nil	Small	Good	Very Good	Complete
Objective 8A	☐	☐	☐	☐	☐
Objective 8B	☐	☐	☐	☐	☐
Objective 8C	☐	☐	☐	☐	☐
Objective 8D	☐	☐	☐	☐	☐
Objective 8E	☐	☐	☐	☐	☐
Objective 8F	☐	☐	☐	☐	☐

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 8.