THE UNIVERSITY OF SYDNEY

MATH1902 Linear Algebra (advanced)

Exercises for Week 9 (beginning May 4) Semester 1 2009

Preliminary Reading:

Chapter 2 of the Linear Algebra book.

Objectives:

By the end of Week 9, to achieve at least a pass level, you should be able to

9A: use row operations to compute the inverse of a matrix,

9B: find the parity of a permutation,

9C: write a permutation in cycle form.

To achieve higher than a pass level you should be able to

9D: relate elementary row operations and elementary matrices,

9E: understand the connection between finding the inverse of a matrix and solving systems of simultaneous linear equations,

9F: compose permutations given either in two-line form or in cycle form.

Preparatory questions. (Answers are on the next page.)

1. Find the inverse of the matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

2. Find the inverse of $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 4 & 2 \end{bmatrix}$, avoiding fractions in your calculations as long as possible.

3. Given the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$,

(i) Determine the parity of σ.

(ii) Write σ as a product of disjoint cycles.

Self-assessment checklist

Tick the box or boxes and seek help from your tutor, if required.

☐ I was unable to complete the Preparatory Questions.

☐ I completed the Preparatory Questions:
☐ with ease. ☐ with some effort. ☐ with difficulty.

Practice questions

4. Let $A = \begin{bmatrix} 1 & -2 & -1 \\ -3 & 5 & 1 \\ 10 & -12 & 8 \end{bmatrix}$.

(i) Use elementary row operations to transform $[\,A \mid I\,]$ to a reduced echelon matrix $[\,I \mid B\,]$, and hence show that A is invertible and find its inverse. Record the row operations you use, and make use of this information to express the matrix A as a product of elementary matrices.

(*ii*) Use the inverse of A (calculated in (*i*)) to solve the following matrix equations for X, Y and Z:

$$AX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad AY = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 4 & 3 \end{bmatrix}, \qquad ZA = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}.$$

5. After applying the following sequence of elementary row operations: $R_1 \leftrightarrow R_3$, $R_2 := R_2 - 2R_1$, $R_3 := R_3 - R_2$, $R_3 := -R_3$, $R_2 := R_2 - 2R_3$, $R_1 := R_1 + 7R_3$, $R_1 := R_1 - 4R_2$ to a certain 3×3 matrix A, the resulting matrix was the identity.

(*i*) Write down the elementary matrices corresponding to the given row operations.

(*ii*) Write down the inverses of the elementary matrices in (*i*).

(*iii*) Write down expressions for A and A^{-1} as products of the elementary matrices from (*i*) or (*ii*) and hence calculate A and A^{-1}.

6. Consider the two systems of linear equations:

$$\begin{aligned} x + 2y + 2z &= 3 \\ y + 3z &= -1 \\ z &= 4 \end{aligned} \qquad\qquad \begin{aligned} x + 2y + 2z &= 2 \\ y + 3z &= 2 \\ z &= 2 \end{aligned}$$

Observe that they have the same coefficient matrix. Solve them both at once by applying row operations to the augmented matrix

$$\left[\begin{array}{ccc|cc} 1 & 2 & 2 & 3 & 2 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right].$$

7. Given a certain 3×3 matrix A and the three systems of equations

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Suppose that the first system has solution $x = 1$, $y = 1$, $z = 0$, the second has solution $x = 0$, $y = 1$, $z = 3$, and the third has solution $x = y = z = 1$. What is the inverse of the matrix A?

8. (*i*) Let A be an 3×3 matrix, and suppose that $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} A = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. Show that A is not invertible. (Hint: suppose that A^{-1} exists, and obtain a contradiction by considering $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} AA^{-1}$.)

(*ii*) Let A be an $n \times n$ matrix, and suppose that there exists a nonzero $1 \times n$ matrix (row vector) B such that $BA = \mathbf{0}$ (where $\mathbf{0}$ is the $1 \times n$ zero matrix). Show that A is not invertible. Similarly, show that if there is a nonzero $n \times 1$ matrix (column vector) C such that AC is zero then A is not invertible.

9. Draw diagrams for the following permutations of 1, 2, 3, 4, 5, 6, 7, and hence determine whether they are odd or even.

(*i*) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 6 & 7 & 4 & 5 \end{pmatrix}$ (*ii*) $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 7 & 4 & 5 & 6 & 2 \end{pmatrix}$

(*iii*) $\sigma\tau$ (*iv*) $\tau\sigma$

10. (*i*) Calculate the product of the following two matrices:

$$A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & p & q \\ 0 & 1 & r & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

What is the inverse of A?

(*ii*) Let A, B, C, D, E, F, G and H be 2×2 matrices, and use them to construct two 4×4 "block matrices" P and Q as follows:

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad Q = \begin{bmatrix} E & F \\ G & H \end{bmatrix}.$$

Show that $PQ = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$.

Answers to Preparatory Questions

1. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

2. The inverse is $\begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} \\ -\frac{5}{3} & -1 & \frac{7}{3} \\ \frac{7}{3} & 2 & -\frac{11}{3} \end{bmatrix}$, as shown by the following row operations

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 := R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & -3 & -2 & 1 & -1 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 := R_3 - 3R_1]{R_2 := R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & -3 & -2 & 1 & -1 & 0 \\ 0 & 11 & 7 & -2 & 3 & 0 \\ 0 & 13 & 8 & -3 & 3 & 1 \end{array}\right] \xrightarrow{R_3 := R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & -3 & -2 & 1 & -1 & 0 \\ 0 & 11 & 7 & -2 & 3 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 := R_2 - 5R_3} \left[\begin{array}{ccc|ccc} 1 & -3 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & 3 & -5 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_3 := R_3 - 2R_2]{R_1 := R_1 + 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 10 & 8 & -15 \\ 0 & 1 & 2 & 3 & 3 & -5 \\ 0 & 0 & -3 & -7 & -6 & 11 \end{array}\right]$$

$$\xrightarrow[R_3 := (-\frac{1}{3})R_3]{\substack{R_1 := R_1 + \frac{4}{3}R_3 \\ R_2 := R_2 + \frac{2}{3}R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{5}{3} & -1 & \frac{7}{3} \\ 0 & 0 & 1 & \frac{7}{3} & 2 & \frac{11}{3} \end{array}\right]$$

3. (*i*) odd (*ii*) $(1, 5, 3)(2, 4)$

Self-assessment checklist:

Think about the work you have completed and how it relates to the objectives on the first page. This is aimed at helping you focus on how well you are going and on the areas in which you may need to do further practice or seek assistance.

In the following table, each row corresponds to one of the objectives listed on the first page. Tick the box corresponding to the level of understanding you believe you have achieved.

My understanding is:	Nil	Small	Good	Very Good	Complete
Objective 9A	☐	☐	☐	☐	☐
Objective 9B	☐	☐	☐	☐	☐
Objective 9C	☐	☐	☐	☐	☐
Objective 9D	☐	☐	☐	☐	☐
Objective 9E	☐	☐	☐	☐	☐
Objective 9F	☐	☐	☐	☐	☐

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 9.