

Preliminary Reading:

Chapter 3 of the Linear Algebra book.

Objectives:

By the end of Week 10, to achieve at least a pass level, you should be able to

10A: use row operations to solve homogeneous equations (revision),

10B: find the rank of a matrix,

10C: calculate the determinant of a matrix.

To achieve higher than a pass level you should be able to

10D: compose permutations given either in two-line form or in cycle form,

10E: calculate determinants using row expansions, elementary row operations and the formula involving parity of permutations.

Preparatory questions. (Answers are on the next page.)

1. Solve the following system of homogeneous linear equations:

$$\begin{aligned}5x + 9y + 3z &= 0 \\ -3x + 5y + 6z &= 0 \\ -x - 5y - 3z &= 0\end{aligned}$$

2. What is the rank of the coefficient matrix of question 1?
3. What is the determinant of the coefficient matrix of question 1?
4. Compute the determinant of $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Self-assessment checklist

Tick the box or boxes and seek help from your tutor, if required.

- I was unable to complete the Preparatory Questions.
- I completed the Preparatory Questions:
- with ease. with some effort. with difficulty.

Practice questions

5. Let $f(x_1, x_2, x_3, x_4)$ be the expression which is the product of all possible factors $x_r - x_s$ with $1 \leq r < s \leq 4$.

(i) Write $f(x_1, x_2, x_3, x_4)$ as a product of linear factors *without* expanding it.

(ii) Express $f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$ in terms of $f(x_1, x_2, x_3, x_4)$ when

(a) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$, (b) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$.

6. For the permutations

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

of $\{1, 2, 3, 4\}$, calculate $\rho \circ \sigma$, $\sigma \circ \rho$ and $\tau \circ (\sigma \circ \rho)$. For each of these permutations, determine whether it is odd or even.

7. Let A be an $n \times n$ matrix. Show that if the determinant of A is nonzero then $\mathbf{x} = \mathbf{0}$ is the only solution of the system of equations $A\mathbf{x} = \mathbf{0}$. By considering what happens when row operations are applied to obtain an equivalent system whose coefficient matrix is in echelon form, show that if the determinant of A is zero then $A\mathbf{x} = \mathbf{0}$ has nonzero solutions.

8. (i) Show that if A is an $n \times n$ matrix and t a scalar then $\det(tA) = t^n \det A$. Deduce that $\det(-A) = -\det A$ if n is odd, while $\det(-A) = \det A$ if n is even.

(ii) Show that $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \det \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$. (Write out all 6 terms.)

(iii) Apply the result of Part (ii) to the matrix $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$, and then, using Part (i), deduce that $\det A = 0$.

9. Let A and B be the following 3×3 matrices: $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$.

Calculate the determinants of both A and B

(i) by using the inductive definition of determinants (the “first row expansion” method), and then

(ii) by using the non-inductive formula for determinants, and then

(iii) by using the row operations method.

Answers to Preparatory Questions

1. We carry out row operations on the coefficient matrix

$$\begin{aligned} & \left[\begin{array}{ccc|c} 5 & 9 & 3 & 0 \\ -3 & 5 & 6 & 0 \\ -1 & -5 & -3 & 0 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & -5 & -3 & 0 \\ -3 & 5 & 6 & 0 \\ 5 & 9 & 3 & 0 \end{array} \right] \\ & \xrightarrow{\substack{R_2 := R_2 - 3R_1 \\ R_3 := R_3 + 5R_1}} \left[\begin{array}{ccc|c} -1 & -5 & -3 & 0 \\ 0 & 20 & 15 & 0 \\ 0 & -16 & -12 & 0 \end{array} \right] \xrightarrow{\substack{R_2 := \frac{1}{20}R_2 \\ R_3 := -\frac{1}{16}R_3}} \left[\begin{array}{ccc|c} -1 & -5 & -3 & 0 \\ 0 & 1 & \frac{3}{4} & 0 \\ 0 & 1 & \frac{3}{4} & 0 \end{array} \right] \\ & \xrightarrow{\substack{R_2 := R_2 - 3R_1 \\ R_3 := R_3 + 5R_1}} \left[\begin{array}{ccc|c} -1 & -5 & -3 & 0 \\ 0 & 1 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

We see that the solution has one parameter, say t . To avoid fractions, set $z = 4t$ so that $y = -3t$ and $x = 3t$.

2. An echelon form of the matrix has two non-zero rows; thus the rank is 2.
3. The coefficient matrix is equivalent to one with a row of zeros; thus the determinant is 0.
4. On expanding along the first row we find that the determinant is

$$0 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 2.$$

Self-assessment checklist:

Think about the work you have completed and how it relates to the objectives on the first page. This is aimed at helping you focus on how well you are going and on the areas in which you may need to do further practice or seek assistance.

In the following table, each row corresponds to one of the objectives listed on the first page. Tick the box corresponding to the level of understanding you believe you have achieved.

My understanding is:	Nil	Small	Good	Very Good	Complete
Objective 10A	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Objective 10B	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Objective 10C	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Objective 10D	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>
Objective 10E	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>	<input type="checkbox"/>

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

www.maths.usyd.edu.au/u/UG/JM/MATH1902/

and then do the Web Quiz for Week 10.