The University of Sydney

MATH1902 Linear Algebra (advanced)

Exercises for Week 11 (beginning May 18) — Semester 1 2009

**Preliminary Reading:**

Chapter 3 of the Linear Algebra book.

**Objectives:**

By the end of Week 11, to achieve at least a pass level, you should be able to

11A: write an invertible matrix as a product of elementary matrices,

11B: calculate the determinant of a matrix,

11C: recognise an eigenvector and find the associated eigenvalue of a matrix.

To achieve higher than a pass level you should be able to

11D: carry out calculations with general $m \times n$ matrices,

11E: use the sigma notation and the Kronecker delta.

**Preparatory questions.** (Answers are on the next page.)

**1.** Show that $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\begin{bmatrix} 2 & 1 & 2 \\ 3 & -4 & 4 \\ 1 & 3 & 5 \end{bmatrix}$, and determine the corresponding eigenvalue.

**2.** Write the matrix $A = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 2 & -3 \\ 2 & 4 & 10 \end{bmatrix}$ as a product of elementary matrices. Write down the determinant of each of the elementary matrices in this product, and hence write down the determinant of $A$.

**Self-assessment checklist**

Tick the box or boxes and seek help from your tutor, if required.

☐ I was unable to complete the Preparatory Questions.

☐ I completed the Preparatory Questions:
☐ with ease. ☐ with some effort. ☐ with difficulty.

**Practice questions**

**3.** Show that $\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$ is an eigenvector for $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & 5 \\ 6 & 10 & 8 \end{bmatrix}$, and determine the corresponding eigenvalue.

**4.** Suppose that $\mathbf{v}$ is an eigenvector for the $n \times n$ matrix $A$ with eigenvalue $\lambda$.

$(i)$ Show that $\mathbf{v}$ is also an eigenvector for $A^2$ and determine the corresponding eigenvalue.

$(ii)$ Assuming that $A$ is invertible, show that $\mathbf{v}$ is also an eigenvector for $A^{-1}$ and determine the corresponding eigenvalue.

**5.** Let $A$ be an $m \times n$ matrix, where $m > n$, and let $D$ be a row echelon matrix obtained from $A$ by applying elementary row operations. Show that $D$ has a row of zeros, and note as a consequence that there is an invertible $m \times m$ matrix $B$ such that $BA$ has a row of zeros. Deduce that it is impossible to find an $n \times m$ matrix $C$ such that $AC = I_m$.

**6.** Find the general solution of the matrix equation $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = X \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

**7.** Let $A$ be an $r \times n$ matrix. For each value of $i$ from 1 to $n$, let $\mathbf{e}_k$ be the $k$-th column of the $n \times n$ identity matrix.

($i$) Using the Kronecker delta, write down the formula for the $j$-th entry of $\mathbf{e}_k$ (where $j$ and $k$ are arbitrary numbers in $\{1, 2, \ldots, n\}$).

($ii$) Using sigma notation, write down the expression for the $i$-th entry of the column vector $A\mathbf{e}_k$ (for arbitrary $i$ in $\{1, 2, \ldots, r\}$ and $k$ in $\{1, 2, \ldots, n\}$), and hence show that $A\mathbf{e}_k$ is the $k$-th column of $A$.

**8.** Use row operations to calculate the determinant of the matrix

$$\begin{bmatrix} x & 0 & 0 & 0 & 0 & a_0 \\ -1 & x & 0 & 0 & 0 & a_1 \\ 0 & -1 & x & 0 & 0 & a_2 \\ 0 & 0 & -1 & x & 0 & a_3 \\ 0 & 0 & 0 & -1 & x & a_4 \\ 0 & 0 & 0 & 0 & -1 & x + a_5 \end{bmatrix}$$

## Answers to Preparatory Questions

**1.** $\begin{bmatrix} 2 & 1 & 2 \\ 3 & -4 & 4 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+1+4 \\ 3-4+8 \\ 1+3+10 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. So the given vector is an eigenvector with eigenvalue 7.

**2.** Apply row operations to $A$ reduce it to $I$:

$$\begin{bmatrix} 1 & 1 & 5 \\ 2 & 2 & -3 \\ 2 & 4 & 10 \end{bmatrix} \xrightarrow[R_3:=R_3-2R_1]{R_2:=R_2-2R_1} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & -13 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & -13 \end{bmatrix}$$

$$\xrightarrow[R_3:=-\frac{1}{13}R_3]{R_2:=\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1:=R_1-R_2]{R_1:=R_1-5R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now $A$ is the product of the elementary matrices corresponding to the inverses of the elementary row operations used, in the order they were used. So

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The determinants of these elementary matrices are (respectively) 1, 1, $-1$, 2, $-13$, 1 and 1. And so $\det A = (-1) \times 2 \times (-13) = 26$.

**Self-assessment checklist:**

Think about the work you have completed and how it relates to the objectives on the first page. This is aimed at helping you focus on how well you are going and on the areas in which you may need to do further practice or seek assistance.

In the following table, each row corresponds to one of the objectives listed on the first page. Tick the box corresponding to the level of understanding you believe you have achieved.

| My understanding is: | Nil | Small | Good | Very Good | Complete |
|---|---|---|---|---|---|
| Objective 11A | ☐ | ☐ | ☐ | ☐ | ☐ |
| Objective 11B | ☐ | ☐ | ☐ | ☐ | ☐ |
| Objective 11C | ☐ | ☐ | ☐ | ☐ | ☐ |
| Objective 11D | ☐ | ☐ | ☐ | ☐ | ☐ |
| Objective 11E | ☐ | ☐ | ☐ | ☐ | ☐ |

**Web Quiz**

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 11.