THE UNIVERSITY OF SYDNEY

MATH1902 Linear Algebra (advanced)

Exercises for Week 12 (beginning May 25) Semester 1 2009

Preliminary Reading:

Chapter 3 of the Linear Algebra book.

Objectives:

By the end of Week 12, to achieve at least a pass level, you should be able to

12A: calculate the characteristic equation of a matrix,

12B: calculate the eigenvalues and eigenvectors of 3×3 matrices.

To achieve higher than a pass level you should be able to

12C: work with matrix equations involving inverses and adjoints,

12D: carry out symbolic calculations with eigenvalues and eigenvectors.

Preparatory questions. (Answers are on the next page.)

1. For which values of λ is the following matrix not invertible:

$$\begin{bmatrix} 3-\lambda & -2 \\ -1 & 2-\lambda \end{bmatrix}.$$

2. Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$.

3. For each eigenvalue of the matrix A of the previous question, find a corresponding eigenvector.

Practice questions

4. Find the eigenvalues and corresponding eigenvectors for $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 1 & 13 \\ 0 & 0 & -3 \end{bmatrix}$.

5. [The Cayley-Hamilton Theorem for 2×2 matrices.] Show that the characteristic equation of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$. Show also that A satisfies the matrix equation $A^2 - (a+d)A + (ad-bc)I_2 = 0_2$, where I_2 and 0_2 are the 2×2 identity and zero matrices respectively.

6. Let A and P be $n \times n$ matrices, with P invertible. Show that A and PAP^{-1} have the same characteristic equation. (Use the rule that $\det(XY) = (\det X)(\det Y)$. And note that $\det X \det Y$ always equals $\det Y \det X$, even though XY need not equal YX.)

7. (i) Given a 3×3 matrix A and a 3×1 column vector $\mathbf{b}$, show that

$$(\operatorname{adj} A)\mathbf{b} = \begin{bmatrix} \det(A_1) \\ \det(A_2) \\ \det(A_3) \end{bmatrix}$$

where A_i is the matrix obtained from A by replacing column i by $\mathbf{b}$.

(*ii*) Suppose that A is invertible and then show that the solution to the matrix equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \det(A_1)/\det(A) \\ \det(A_2)/\det(A) \\ \det(A_3)/\det(A) \end{bmatrix}.$$

[This formula for the solution of a system of n equations in n unknowns is known as *Cramer's Rule*. (We have only presented the case $n = 3$ here.) Note that the coefficient matrix must be invertible for Cramer's Rule to apply.]

(*iii*) Use (*ii*) to solve the following equations:

$$\begin{aligned} x + 2y + 2z &= 5 \\ x + 3y + z &= 0 \\ x + 3y + 2z &= -2 \end{aligned}$$

Which method of solving equations do you prefer: using row operations or Cramer's rule?

8. The *Hessian* of a function $u(x_1, x_2)$ of two variables is the determinant of the 2×2 matrix whose (i, j)-th entry is $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$. Find the Hessian of $ax_1^2 + bx_1x_2 + cx_2^2$.

Answers to Preparatory Questions

1. The determinant of the given matrix is $(\lambda - 1)(\lambda - 4)$. This is 0 when $\lambda = 1$ or $\lambda = 4$ and so these are the values for which the matrix is not invertible.

2. The calculation of the preceding exercise shows that the eigenvalues of A are 1 and 4.

3. When $\lambda = 1$ an eigenvector is any non-zero multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. When $\lambda = 4$ an eigenvector is any non-zero multiple of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 12.