THE UNIVERSITY OF SYDNEY

MATH1902 Linear Algebra (advanced)

Exercises for Week 13 (beginning June 1) Semester 1 2009

Preliminary Reading:

Chapter 3 of the Linear Algebra book.

Objectives:

By the end of Week 13, to achieve at least a pass level, you should be able to

13A: calculate the characteristic equation of a matrix,

13B: calculate the eigenvalues and eigenvectors of 3×3 matrices.

To achieve higher than a pass level you should be able to

13C: diagonalize a matrix,

13D: determine when column vectors are linearly independent.

Preparatory questions. (Answers are on the next page.)

1. Let $A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & -2 & 2 \\ 3 & -6 & 5 \end{bmatrix}$.

(i) Find the characteristic polynomial of A.

(ii) Find the determinant of A.

(iii) The sum of the diagonal elements of A is one of the coefficients of the characteristic polynomial. Which one?

2. Let A be an $n \times n$ matrix and B an $n \times r$ matrix. Suppose that $AB = \lambda B$ for some scalar λ. Show that each column of B is either zero or an eigenvector of A.

3. Suppose that A is a 3×3 matrix such that the sum of the columns of A is $\mathbf{0}$. Write down an eigenvector and the corresponding eigenvalue for A.

Practice questions

4. In each case find the eigenvalues and corresponding eigenvectors for the matrix A, and hence find a matrix M such that $M^{-1}AM$ is diagonal.

(i) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. ($ii$) $A = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

5. (i) Show that the characteristic polynomials of the matrices $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ are both equal to $(x-1)^3(x-5)$.

(ii) Find an invertible matrix T such that $T^{-1}AT$ is diagonal.

(iii) Show that there is no invertible matrix U such that $U^{-1}BU$ is diagonal.

6. Let A be an $n \times n$ matrix and $\mathbf{u}$ a $1 \times n$ matrix. (That is, $\mathbf{u}$ is a row vector.) Show that if $\mathbf{u}A = \lambda\mathbf{u}$ for some scalar λ then either $\mathbf{u}$ is zero or λ is a root of the characteristic polynomial of A.

7. Let A be a 3×3 matrix. Show that in the first row expansion of $\det(A - xI)$, only the first term involves x^2 and x^3, and hence check that the coefficient of x^2 is $a_{11} + a_{22} + a_{33}$. Similarly check that when A is 4×4 the coefficient of x^3 is $-\sum_{i=1}^{4} a_{ii}$. Generalise this to the $n \times n$ case. (The sum of the entries on the main diagonal of a square matrix is called the *trace* of the matrix. It equals the sum of the eigenvalues, counted according to their multiplicities as roots of the characteristic polynomial.)

8. Let $M = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$. Show that each of the columns of M is an eigenvector for the matrix $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$. By considering its determinant, show that M is invertible. Without calculating M^{-1}, show that $M^{-1}AM$ is a diagonal matrix, and evaluate it.

9. Let $\mathbf{u}$ be a nonzero n-component row vector and $\mathbf{v}$ a nonzero n-component column vector, and let A be the $n \times n$ matrix $\mathbf{vu}$.

(*i*) Show that $\mathbf{v}$ is an eigenvector for A corresponding to the eigenvalue $\mathbf{uv}$.

(*ii*) Show that every nonzero n-component column vector $\mathbf{w}$ such that $\mathbf{uw} = 0$ is an eigenvector for A with eigenvalue 0.

(*iii*) By taking $\mathbf{u} = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$, use the results of Parts (*i*) and (*ii*) to find a (-1)-eigenvector and two 0-eigenvectors for the matrix $\begin{bmatrix} 4 & 8 & -4 \\ -1 & -2 & 1 \\ 3 & 6 & -3 \end{bmatrix}$, such that the 0-eigenvectors are not scalar multiples of each other.

Hence find an invertible matrix M such that $AM = M \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Answers to Preparatory Questions

1. (*i*) We have

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 & 1 \\ 2 & -2-\lambda & 2 \\ 3 & -6 & 5-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} -2-\lambda & 2 \\ -6 & 5-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 3 & 5-\lambda \end{vmatrix} + \begin{vmatrix} 2 & -2-\lambda \\ 3 & -6 \end{vmatrix}$$

$$= (3-\lambda)((-2-\lambda)(5-\lambda) + 12) + 2(10 - 2\lambda - 6) + (-12 + 6 + 3\lambda)$$

$$= 8 - 12\lambda + 6\lambda^2 - \lambda^3$$

$$= -(\lambda - 2)^3$$

(*ii*) The determinant is the value of the characteristic polynomial when $\lambda = 0$, namely 8.

(*iii*) The sum of the diagonal elements is 6, the coefficient of λ^2. (It is always true that for an $n \times n$ matrix the sum of the diagonal elements is the coefficient of $(-\lambda)^{n-1}$ in the characteristic polynomial. See exercise 7.)

2. $AB = \lambda B$ gives $A\mathbf{b} = \lambda\mathbf{b}$ for each column $\mathbf{b}$ of the matrix B. If $\mathbf{b}$ is nonzero then by definition this column is a λ-eigenvector of A.

3. To say that the sum of the columns of A is $\mathbf{0}$ is to say that their linear combination with coefficients all 1 is $\mathbf{0}$. That is $A\begin{bmatrix}1\\1\\1\end{bmatrix} = \mathbf{0} = 0\begin{bmatrix}1\\1\\1\end{bmatrix}$ and so $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 0.

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 13.