The University of Sydney

MATH1902 Linear Algebra (advanced)

Exercises for Week 1 (beginning March 2) Semester 1 2009

Reading:

Chapter 1 of the Vectors book.

Objectives:

By the end of Week 1, to achieve at least a pass level, you should

1A: be able to identify and distinguish between scalar and vector quantities.

1B: know what is meant by the position vector of a point and illustrate this with a diagram.

1C: be able to explain how to add vectors using either the triangle rule or the parallelogram rule.

To achieve higher than a pass level you should

1D: know what it means for a point to divide a line segment in a given ratio.

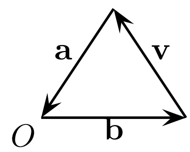
1E: be able to use position vectors to solve problems in geometry.

1F: be able to use the laws of vector algebra to simplify expressions involving vectors and scalars

Preparatory questions. (Answers are on the next page.)

1. Classify the following as scalar or vector quantities: (*i*) temperature, (*ii*) force.

2. In the diagram below, express the vector $\mathbf{v}$ in terms of the vectors $\mathbf{a}$ and $\mathbf{b}$.

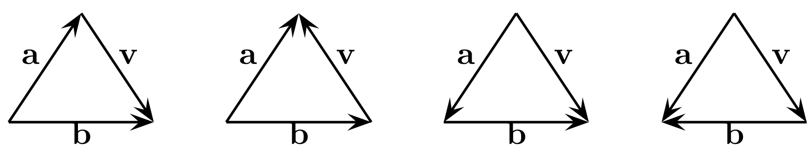


3. In the diagram above, mark the point whose position vector relative to O is $\frac{1}{2}(\mathbf{b} - \mathbf{a})$.

4. If $|\mathbf{v}| = 2$ and if $\mathbf{u} = -3\mathbf{v}$, what is the magnitude of $\mathbf{u} + \mathbf{v}$?

Practice questions

5. For each diagram below, express $\mathbf{v}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.



Solution.

The answers (in order) are: $\mathbf{v} = -\mathbf{a} + \mathbf{b}$, $\mathbf{v} = \mathbf{a} - \mathbf{b}$, $\mathbf{v} = \mathbf{a} + \mathbf{b}$, $\mathbf{v} = \mathbf{a} - \mathbf{b}$.

6. Simplify $3\mathbf{a} + 2\mathbf{b} - 6\mathbf{c} - 5(-\frac{1}{2}(\mathbf{c} + \mathbf{a}) + 3(\mathbf{c} - \frac{1}{2}\mathbf{b}))$.

Solution.

$\frac{11}{2}\mathbf{a} + \frac{19}{2}\mathbf{b} - \frac{37}{2}\mathbf{c}$.

7. Let $ABCDEF$ be a regular hexagon and denote $\vec{AB}$ by $\mathbf{a}$ and $\vec{CD}$ by $\mathbf{c}$. Find vector expressions in terms of $\mathbf{a}$ and $\mathbf{c}$ for the vectors $\vec{BC}$, $\vec{DE}$, $\vec{EF}$ and $\vec{FA}$.

Solution.

Use the fact that opposite sides are parallel and that if O is the centre of the hexagon, then the triangles OAB, OBC, $\ldots$, OFA are equilateral. Then $ABCO$ and $BODC$ are parallelograms and hence $\vec{BO} = \mathbf{c}$ and $\vec{BC} = \vec{AO} = \vec{AB} + \vec{BO} = \mathbf{a} + \mathbf{c}$. Also, $\vec{DE} = -\mathbf{a}$, $\vec{EF} = -(\mathbf{a} + \mathbf{c})$, and $\vec{FA} = -\mathbf{c}$.

8. (Group discussion) Suppose that $ABCDEF$ is a regular hexagon. Is it true that $\vec{AC} = \vec{FD}$? What does this mean?

Solution.

Continuing the previous exercise, we have $\vec{AC} = \vec{AB} + \vec{BC} = 2\mathbf{a} + \mathbf{c}$ and also $\vec{FD} = \vec{FE} + \vec{ED} = -\vec{EF} - \vec{DE} = 2\mathbf{a} + \mathbf{c} = \vec{AB}$. So the statement is true. What it means is that the line from A to C and the line from F to D have the same length and point in the same direction. Equivalently, the quadrilateral $ACDF$ is a parallelogram.

9. (Group discussion) What is the difference (if any) between the *triangle* and the *parallelogram* rules of addition?

Solution.

An obvious difference is that in the parallelogram rule, the vectors originate from the same point, whereas in the triangle rule the head of one vector is at the tail of the next.

10. (Group discussion) Does your concept of vector allow for the zero vector? If not, how would you fix it?

Solution.

According to the definition given on p.3 of the book, a vector consists of a magnitude and a direction. Moreover, according to p.10, vectors $\mathbf{u}$ and $\mathbf{v}$ are equal if and only if they have the same magnitude and the same direction. However, this latter rule is not correct as it stands: we should add the words "except that $\mathbf{u} = \mathbf{v}$ whenever $\mathbf{u}$ and $\mathbf{v}$ both have magnitude zero." The zero vector in any direction is equal to the zero vector in any other direction.

There is another approach to vectors that is more general than ours. In this more general treatment one lists a set of algebraic properties that vectors should satisfy, and defines a *vector space* to be any set of things that satisfy these properties. These *vector space axioms* are, in fact, just the properties of vector addition and scalar multiplication listed on p.13 of the book. Roughly speaking, a vector space is any set of things for which addition and multiplication can be defined in a sensible way. There are many different examples of vector spaces, and hence many different kinds of vectors. The concepts of magnitude and direction are special to the vector space that we have been discussing (but every vector space must have a zero vector).

11. Given points P and Q that divide the sides AB and AC of the triangle ABC in the ratios $\lambda : (1-\lambda)$ and $\mu : (1-\mu)$, show that if $\vec{PQ} = \nu\vec{BC}$, then $\lambda = \mu = \nu$.

Solution.

Given any point O we have, using the division ratio formula,

$$\begin{aligned}
\vec{OP} &= (1-\lambda)\vec{OA} + \lambda\vec{OB} \\
\vec{OQ} &= (1-\mu)\vec{OA} + \mu\vec{OC} \text{ and} \\
\vec{PQ} &= \vec{OQ} - \vec{OP} = (\lambda-\mu)\vec{OA} - \lambda\vec{OB} + \mu\vec{OC}
\end{aligned}$$

Equating this to $\nu\vec{BC} = \nu\vec{OC} - \nu\vec{OB}$ we find that

$$\nu\vec{OC} - \nu\vec{OB} = (\lambda-\mu)\vec{OA} - \lambda\vec{OB} + \mu\vec{OC}.$$

On putting $O = A$ this simplifies to $(\nu-\mu)\vec{AC} = (\nu-\lambda)\vec{AB}$. But $\vec{AB}$ and $\vec{AC}$ have different directions and so the only way this can happen is for the coefficients to be 0. That is $\nu = \mu$ and $\nu = \lambda$.

12. Given a plane quadrilateral $ABCD$, suppose that the diagonals AC and BD intersect at P and the sides AB and CD intersect at Q. If Q divides AB in the ratio $3 : -1$ and if it divides CD in the ratio $-5 : 7$, show that P divides AC in the ratio $7 : 1$ and BD in the ratio $5 : 3$.

Solution.

We have $\vec{OQ} = \frac{-\vec{OA}+3\vec{OB}}{2}$ and $\vec{OQ} = \frac{7\vec{OC}-5\vec{OD}}{2}$. Equating these expressions and multiplying by 2 yields

$$-\vec{OA} + 3\vec{OB} = 7\vec{OC} - 5\vec{OD}$$

and this can be rewritten (after dividing by 8) as

$$\frac{\vec{OA} + 7\vec{OC}}{8} = \frac{3\vec{OB} + 5\vec{OD}}{8}.$$

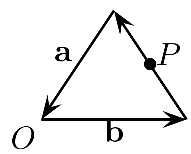
We see that this is the position vector of a point on both AC and on BD, i.e., it must be P and therefore P divides AC in the ratio $7 : 1$ and BD in the ratio $5 : 3$.

Answers to Preparatory Questions

1. (i) scalar (ii) vector

2. $\mathbf{v} = -\mathbf{a} - \mathbf{b}$.

3.



4. 4

Web Quiz

There are additional self assessment tasks on the Web. Follow the links from the School of Mathematics and Statistics main page `http://www.maths.usyd.edu.au` to MATH1902 and then do the Web Quiz for Week 1.