The University of Sydney
MATH1902 Linear Algebra (advanced)

Exercises for Week 2 (beginning March 9) Semester 1 2009

Reading:

Chapter 2 of the Vectors book.

Objectives:

By the end of Week 2, to achieve at least a pass level, you should

2A: be able to find the Cartesian coordinate form of position vectors in two and three dimensions.

2B: be able to find the polar form of a vector given in Cartesian form.

2C: be able to find the Cartesian form of a vector given in polar form.

2D: be able to find the coordinates of the point that divides a line segment in a given ratio.

To achieve higher than a pass level you should

2E: be able to translate problems expressed in words into vector notation.

2F: be able to reason logically with vectors in order to prove theorems in geometry.

2G: practice visualising vector figures in two and three dimensions.

2H: be able to apply the concept of linear independence to collections of vectors in two and three dimensions.

Preparatory questions. (Answers are on the next page.)

1. Given points A and B in space (with origin at O) with Cartesian coordinates $(2, -1, 3)$ and $(-1, 1, 0)$ respectively.

(i) Write the position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in Cartesian form.

(ii) Write $\overrightarrow{AB}$ in Cartesian form.

(iii) Find the length of $\overrightarrow{AB}$.

(iv) Find the Cartesian coordinates of the point P that divides AB in the ratio $3 : 4$.

2. Draw a diagram showing the vector $\mathbf{v} = 4(\cos\pi/3\,\mathbf{i} + \sin\pi/3\,\mathbf{j})$ and find the coordinates of the point P such that $\overrightarrow{OP} = \mathbf{v}$.

3. Given the vectors $\mathbf{a} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = 2\mathbf{j} - \mathbf{k}$, evaluate

(i) $2\mathbf{a} - \mathbf{b} - 3\mathbf{c}$ (ii) $|\mathbf{a}|$ (iii) $\hat{\mathbf{a}}$.

Practice questions

4. If $\mathbf{A} = -12\mathbf{i} + 4\mathbf{j}$, find A, $\hat{\mathbf{A}}$ and the polar form of $\mathbf{A}$. (To find the polar form find r and θ and express $\mathbf{A}$ in the form $r(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$.)

Solution.

$A = \sqrt{(-12)^2 + 4^2} = 4\sqrt{10}$, and $\hat{\mathbf{A}} = (1/A)\mathbf{A} = (-3\mathbf{i}+\mathbf{j})/\sqrt{10}$. If we write $\mathbf{A}$ in polar form as $r(\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}$ then r is length of $\mathbf{A}$. Thus $r = A = 4\sqrt{10}$, and we must find θ such that $\mathbf{A} = 4\sqrt{10}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$. That is, $(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = \hat{\mathbf{A}}$. So $\cos\theta = -3/\sqrt{10}$ and $\sin\theta = 1/\sqrt{10}$. This means that θ is in the second quadrant, and $\theta = \cos^{-1}(-3/\sqrt{10}) \approx 2.8198$ radians. (Or, using the tangent, we can say that θ is the second quadrant angle $\pi - \tan^{-1}(1/3)$.)

5. Let $\mathbf{i}$ and $\mathbf{j}$ denote displacements of 1 km east and north, respectively. An aircraft flies 300 km southeast and then 150 km in the direction 30° west of north. Find

(i) the above displacements and their vector sum in terms of $\mathbf{i}$ and $\mathbf{j}$.

(ii) the final distance and direction of the aircraft from its starting position.

Solution.

(i) The displacement for the first leg of the journey is

$$\mathbf{u} = 300\cos(\pi/4)\mathbf{i} - 300\sin(\pi/4)\mathbf{j} = 150\sqrt{2}(\mathbf{i} - \mathbf{j}).$$

The displacement for the second leg is

$$\mathbf{v} = -150\sin(\pi/6)\mathbf{i} + 150\cos(\pi/6)\mathbf{j} = 75(-\mathbf{i} + \sqrt{3}\,\mathbf{j}).$$

(ii) The total displacement is $\mathbf{u} + \mathbf{v}$. Using a calculator (or otherwise) we see that this is (approx.) $137.13\,\mathbf{i} - 82.23\,\mathbf{j}$. The length of this vector is 159.90 and if its angle below $\mathbf{i}$ is θ, then $\cos\theta \approx 0.8576$ (or $\tan\theta \approx 0.5996$), hence $\theta \approx 0.54$ radians ($\approx 30.95°$) south of east.

6. Suppose that no two of the points A, B, C are coincident. Show that these three points will be collinear if and only if there exist three non–zero real numbers α, β, γ such that

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}, \quad \alpha + \beta + \gamma = 0. \tag{1}$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of A, B and C respectively.

Solution.

(If): If equations (1) hold, we may divide by $\gamma = -(\alpha + \beta) \neq 0$ to obtain

$$\frac{\alpha\mathbf{a} + \beta\mathbf{b}}{-(\alpha + \beta)} + \mathbf{c} = \mathbf{0}, \qquad \text{i.e.,} \qquad \mathbf{c} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta},$$

showing that C, A and B are collinear (in fact, C divides AB in the ratio $\beta : \alpha$).

(Only if): Conversely, if A, B, C are collinear and no two points are coincident, C must divide AB in some ratio, which may be taken as $\beta : \alpha$, where α and β are nonzero. Thus $\mathbf{c} = \frac{\alpha\mathbf{a}+\beta\mathbf{b}}{\alpha+\beta}$, or $\alpha\mathbf{a} + \beta\mathbf{b} - (\alpha + \beta)\mathbf{c} = \mathbf{0}$. Set $\gamma = -(\alpha + \beta)$. Then we have $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$ and $\alpha + \beta + \gamma = 0$.

Note that if equations (1) hold, and one of α, β, γ is zero (and the other two nonzero), then two of the points must be coincident. Eg, if $\gamma = 0$, we have $\alpha+\beta = 0$ and $\alpha\mathbf{a}+\beta\mathbf{b} = \mathbf{0}$, giving $\beta = -\alpha$ and $\alpha\mathbf{a}-\alpha\mathbf{b} = \mathbf{0}$. Now assuming that

$\alpha \neq 0$, we must have $\mathbf{a} = \mathbf{b}$, implying that A and B are coincident. (Obviously, if all three of α, β and γ are zero then the equations (1) give no information.)

7. (*i*) Suppose that no three of the points A, B, C, D are collinear. Show that these four points will be coplanar (lie in a plane) if and only if there exist four nonzero real numbers α, β, γ, δ, such that

$$(2) \qquad \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} + \delta\mathbf{d} = \mathbf{0}, \quad \alpha + \beta + \gamma + \delta = 0.$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are the position vectors of A, B, C and D respectively.

(*ii*) Using the result of part (*i*), show if D is a point in the plane of the triangle ABC, and the lines AD, BD, CD cut the sides opposite A, B, C in the points R, S, T, then R divides BC in the ratio $\gamma : \beta$. Find the corresponding ratios in which S and T cut CA and AB.

(*iii*) Hence show that the product of the ratios in which R, S and T divide BC, CA and AB is 1. (Theorem of Ceva.)

Solution.

(*i*) The assumption that no three of the points are collinear is meant to include the assumtion that the points are all distinct. (If $A = B$, for example, then A, B, C are collinear.) Assume first that the points are coplanar. Since the lines BC and DA lie in this plane, either they are parallel or else they intersect. (Note that lines that do not lie in a common plane do not intersect and are not parallel.) If BC is parallel to DA then $\overrightarrow{BC} = \lambda\overrightarrow{DA}$ for some scalar λ; that is, $\mathbf{c} - \mathbf{b} = \lambda(\mathbf{a} - \mathbf{d})$. This gives $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} + \delta\mathbf{d} = \mathbf{0}$ with α, β, γ, δ respectively equal to $-\lambda$, -1, 1, λ; so $\alpha + \beta + \gamma + \delta = 0$, as required. On the other hand, is BC and DA intersect, at a point P say, then P divides BC in some ratio $r : s$ and DA in some ratio $t : u$. Here r, s, t and u are all nonzero, since P is not equal to A, B, C or D. (If P were equal to A, for example, then A, B and C would be collinear.) This gives $\overrightarrow{OP} = \frac{1}{s+r}(s\mathbf{b} + r\mathbf{c})$ and $\overrightarrow{OP} = \frac{1}{t+u}(t\mathbf{a} + u\mathbf{d})$. So

$$\frac{s}{s+r}\mathbf{b} + \frac{r}{s+r}\mathbf{c} = \frac{t}{t+u}\mathbf{a} + \frac{u}{t+u}\mathbf{d}.$$

Observe that on each side of this equation the sum of the coefficients is 1: for example, $(s/(s+r)) + (r/(s+r) = (s+r)/(s+r) = 1$. (We have incidentally proved the important general fact that the position vectors of all the points on the line AB have the form $\lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ for some λ. The coefficients add up to 1.) Moving all the terms to the left hand side gives an equation of the form $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} + \delta\mathbf{d} = \mathbf{0}$, where the coefficients add up to $1 - 1 = 0$, as required.

Conversely, let us assume that equations (2) hold for nonzero α, β, γ, δ. Then $\alpha + \beta = -(\gamma + \delta)$, and if this quantity is nonzero we can divide the equation $\alpha\mathbf{a} + \beta\mathbf{b} = -(\gamma\mathbf{c} + \delta\mathbf{d})$ through by it and get

$$\frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta} = \frac{-(\gamma\mathbf{c} + \delta\mathbf{d})}{\alpha + \beta} = \frac{-(\gamma\mathbf{c} + \delta\mathbf{d})}{-(\gamma + \delta)} = \frac{\gamma\mathbf{c} + \delta\mathbf{d}}{\gamma + \delta} = \mathbf{q} \quad \text{(say)}.$$

Here $\mathbf{q}$ is the position vector of a point that is on both AB and CD. So these lines—and hence the four points A, B, C and D—are coplanar.

If $\alpha+\beta=0$ (and consequently $\gamma+\delta=0$), we have $\beta=-\alpha$ and $\gamma=-\delta$ and the first equation in (2) becomes $\alpha(\mathbf{a}-\mathbf{b})=\delta(\mathbf{c}-\mathbf{d})$. Since neither α nor δ is zero, we have the result that the lines AB and CD are parallel. This again means that A, B, C, D are coplanar.

(*ii*) Since the four points A, B, C, D are coplanar, equations (2) hold. By the answer to Part (*i*) we also know that $\beta+\gamma\neq 0$, since otherwise AD would be parallel to BC, contrary to our assumption that AD intersects BC. Since $\alpha+\delta=-(\beta+\gamma)$, we also have $\alpha+\delta\neq 0$. So $\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}+\delta\mathbf{d}=\mathbf{0}$ gives

$$\frac{\alpha\mathbf{a}+\delta\mathbf{d}}{\alpha+\delta}=\frac{\beta\mathbf{b}+\gamma\mathbf{c}}{\beta+\gamma}=\mathbf{r},\quad\text{say.}$$

Here $\mathbf{r}$ is the position vector of a point R that divides BC in the ratio $\gamma:\beta$. Similarly

$$\frac{\beta\mathbf{b}+\delta\mathbf{d}}{\beta+\delta}=\frac{\gamma\mathbf{c}+\alpha\mathbf{a}}{\gamma+\alpha}=\mathbf{s},$$

where $\mathbf{s}$ is the position vector of S, a point dividing CA in the ratio $\alpha:\gamma$, and

$$\frac{\gamma\mathbf{c}+\delta\mathbf{d}}{\gamma+\delta}=\frac{\alpha\mathbf{a}+\beta\mathbf{b}}{\alpha+\beta}=\mathbf{t},$$

where T divides AB in the ratio $\beta:\alpha$.

(*iii*) From Part (*ii*) the product of the ratios is $(\gamma/\beta)(\alpha/\gamma)(\beta/\alpha)=1$.

8. Find the sum of the three vectors determined by the diagonals of three adjacent faces of a cube passing through a given corner; the vectors being directed away from the corner. (Use Cartesian coordinates. Take the origin to be the given corner, axes along the edges of the cube, and take the edge length to be λ.)

Solution.

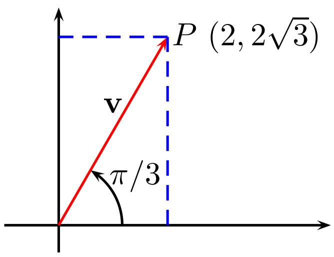
Let O be the origin and take unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ along the edges of the cube. If the edge length is λ, then the diagonals of the three faces that contain O are $\lambda(\mathbf{i}+\mathbf{j})$, $\lambda(\mathbf{j}+\mathbf{k})$ and $\lambda(\mathbf{i}+\mathbf{k})$. Thus the sum of the diagonals is $2\lambda(\mathbf{i}+\mathbf{j}+\mathbf{k})$.

Answers to Preparatory Questions

1. (*i*) $\overrightarrow{OA}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}$, $\overrightarrow{OB}=-\mathbf{i}+\mathbf{j}$. (*ii*) $\overrightarrow{AB}=-3\mathbf{i}+2\mathbf{j}-3\mathbf{k}$.

(*iii*) $|\overrightarrow{AB}|=\sqrt{22}$. (*iv*) $\overrightarrow{OP}=\frac{1}{7}(5\mathbf{i}-\mathbf{j}+12\mathbf{k})$.

2.



3. (*i*) $-7\mathbf{i}-\mathbf{j}-\mathbf{k}$ (*ii*) 3 (*iii*) $-\frac{2}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}-\frac{1}{3}\mathbf{k}$

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 2.