THE UNIVERSITY OF SYDNEY
MATH1902 Linear Algebra (advanced)

Exercises for Week 3 (beginning March 16) Semester 1 2009

**Preliminary Reading:**

Chapter 3 of the Vectors book.

**Objectives:**

By the end of Week 3, to achieve at least a pass level, you should

3A: be able to find the length and direction cosines of a vector in three dimensions

3B: be able to calculate the scalar product of two vectors

3C: be able to calculate the projection of a vector $\mathbf{v}$ in the direction of a vector $\mathbf{u}$

3D: be able to calculate the vector product of two vectors.

To achieve higher than a pass level you should

3F: be able to use scalar and vector products to derive theorems in geometry.

3G: be able to carry out algebraic manipulations involving both scalar and vector products.

**Preparatory questions.** (Answers are on the next page.)

**1.** The vertices of a triangle are the points $A\,(2,-1,-3)$, $B\,(4,2,3)$ and $C\,(6,3,4)$. Find

($i$) the Cartesian coordinates of the vectors $\vec{AB}$ and $\vec{AC}$,

($ii$) the lengths of the vectors found in ($i$),

($iii$) the direction cosines of the vectors found in ($i$).

**2.** Given the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, calculate

($i$) $\mathbf{u} \cdot \mathbf{v}$;

($ii$) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$;

($iii$) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$;

($iv$) the (scalar) component of $\mathbf{v}$ in the direction of $\mathbf{u}$;

($v$) the (vector) projection of $\mathbf{v}$ in the direction of $\mathbf{u}$.

**3.** Given the vectors $\mathbf{u} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, find

($i$) the lengths of $\mathbf{u}$ and $\mathbf{v}$;

($ii$) $\mathbf{u} \times \mathbf{v}$;

($iii$) the sine of the angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$;

($iv$) $(3\mathbf{u} - 2\mathbf{v}) \times (\mathbf{u} + 5\mathbf{v})$.

**Practice questions**

**4.** Show that $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ are perpendicular. Find a vector of unit length that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

*Solution.*

$\mathbf{a}\cdot\mathbf{b} = 10-2-8 = 0$; so $\mathbf{a}\perp\mathbf{b}$. The vector $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. By the formula (p. 52 of the book), $\mathbf{a}\times\mathbf{b} = -6\mathbf{i}+24\mathbf{j}+9\mathbf{k}$. To find a unit vector in the same direction as $\mathbf{a}\times\mathbf{b}$, divide $\mathbf{a}\times\mathbf{b}$ by its own length. Now $|-6\mathbf{i}+24\mathbf{j}+9\mathbf{k}| = \sqrt{6^2+24^2+9^2} = 3\sqrt{2^2+8^2+3^2} = 3\sqrt{77}$, and so the required unit vector is $\mathbf{u} = \dfrac{(-2\mathbf{i}+8\mathbf{j}+3\mathbf{k})}{\sqrt{77}}$. (Note that the negative of this, namely $\frac{2}{\sqrt{77}}\mathbf{i} - \frac{8}{\sqrt{77}}\mathbf{j} - \frac{3}{\sqrt{77}}\mathbf{k}$, is an equally correct answer.)

**5.** (*i*) Show that, for all values of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{h}$,

$$(\mathbf{a}-\mathbf{b})\cdot(\mathbf{h}-\mathbf{c}) + (\mathbf{b}-\mathbf{c})\cdot(\mathbf{h}-\mathbf{a}) + (\mathbf{c}-\mathbf{a})\cdot(\mathbf{h}-\mathbf{b}) = 0.$$

(*ii*) Use this identity to show that the altitudes of any triangle $ABC$ meet in a point $H$. (An *altitude* of the triangle $ABC$ is a line from one of the vertices (say $A$) to a point on the opposite side that is perpendicular to that side (i.e. perpendicular to $BC$). The point $H$ where all three altitudes meet is called the *orthocentre* of the triangle).
[Hint: If two terms of the LHS of the above identity are zero, the third term will also be zero.]

(*iii*) Show that, for all values of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{k}$,

$$(\mathbf{a}-\mathbf{b})\cdot\left(\mathbf{k}-\frac{\mathbf{a}+\mathbf{b}}{2}\right) + (\mathbf{b}-\mathbf{c})\cdot\left(\mathbf{k}-\frac{\mathbf{b}+\mathbf{c}}{2}\right) + (\mathbf{c}-\mathbf{a})\cdot\left(\mathbf{k}-\frac{\mathbf{c}+\mathbf{a}}{2}\right) = 0.$$

(*iv*) Use this identity to show that the perpendicular bisectors of the sides of any triangle $ABC$ meet in a point $K$ (the circumcentre of the triangle).

*Solution.*

(*i*) By direct calculation,

$$\begin{aligned}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{h}-\mathbf{c}) &+ (\mathbf{b}-\mathbf{c})\cdot(\mathbf{h}-\mathbf{a}) + (\mathbf{c}-\mathbf{a})\cdot(\mathbf{h}-\mathbf{b})\\ &= \mathbf{a}\cdot\mathbf{h} - \mathbf{b}\cdot\mathbf{h} - \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{h} - \mathbf{c}\cdot\mathbf{h} - \mathbf{b}\cdot\mathbf{a} + \mathbf{c}\cdot\mathbf{a}\\ &\quad + \mathbf{c}\cdot\mathbf{h} - \mathbf{a}\cdot\mathbf{h} - \mathbf{c}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b}\\ &= 0.\end{aligned}$$

(*ii*) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the position vectors of the points $A$, $B$, $C$ with respect to some origin $O$. Let $\mathbf{h}$ be the position vector of the point $H$ that is the point of intersection of the line from $A$ perpendicular to $BC$ and the line from $B$ perpendicular to $CA$. Then $\overrightarrow{AH} = \mathbf{h}-\mathbf{a}$ is perpendicular to $\overrightarrow{CB} = \mathbf{b}-\mathbf{c}$, and $\overrightarrow{BH} = \mathbf{h}-\mathbf{b}$ is perpendicular to $\overrightarrow{AC} = \mathbf{c}-\mathbf{a}$, giving the two equations

$$(\mathbf{b}-\mathbf{c})\cdot(\mathbf{h}-\mathbf{a}) = 0,$$
$$(\mathbf{c}-\mathbf{a})\cdot(\mathbf{h}-\mathbf{b}) = 0.$$

From the identity in part (*i*) we then obtain $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{h}-\mathbf{c}) = 0$, showing that $CH$ is perpendicular to $AB$. So the altitude from $C$ to $AB$ must coincide with the line $CH$, as required.

($iii$) By direct calculation

$$\begin{aligned}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{k}-\frac{\mathbf{a}+\mathbf{b}}{2})+(\mathbf{b}-\mathbf{c})\cdot(\mathbf{k}-\frac{\mathbf{b}+\mathbf{c}}{2})+(\mathbf{c}-\mathbf{a})\cdot(\mathbf{k}-\frac{\mathbf{c}+\mathbf{a}}{2})\\
&=\mathbf{a}\cdot\mathbf{k}-\mathbf{b}\cdot\mathbf{k}-\frac{1}{2}(\mathbf{a}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b})+\mathbf{b}\cdot\mathbf{k}-\mathbf{c}\cdot\mathbf{k}-\frac{1}{2}(\mathbf{b}\cdot\mathbf{b}-\mathbf{c}\cdot\mathbf{c})\\
&\quad+\mathbf{c}\cdot\mathbf{k}-\mathbf{a}\cdot\mathbf{k}-\frac{1}{2}(\mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{a})\\
&=0.\end{aligned}$$

($iv$) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be the position vectors of the points $A$, $B$, $C$ with respect to some origin $O$. Let $P$, $Q$ and $R$ be the midpoints of $AB$, $BC$ and $CA$, respectively, and let $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ be their position vectors. Then

$$\mathbf{p}=(\mathbf{a}+\mathbf{b})/2,\quad \mathbf{q}=(\mathbf{b}+\mathbf{c})/2,\quad \text{and}\quad \mathbf{r}=(\mathbf{c}+\mathbf{a})/2.$$

Suppose that $\mathbf{k}$ is the position vector of the point $K$ that is the point of intersection of the perpendicular bisectors of the sides $AB$ and $BC$. Then $KP\perp AB$ and $KQ\perp BC$. This gives us the two equations

$$\begin{aligned}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{k}-\mathbf{p})&=0,\\(\mathbf{b}-\mathbf{c})\cdot(\mathbf{k}-\mathbf{q})&=0.\end{aligned}$$

From the identity in Part ($iii$) it follows that $(\mathbf{c}-\mathbf{a})\cdot(\mathbf{k}-\mathbf{r})=0$, from which we conclude that $KR\perp CA$. So the point $K$ is also on the perpendicular bisector of $BC$, which gives us the desired result.

**6.** Consider the vector triple product $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$.

($i$) For any vectors $\mathbf{U}$ and $\mathbf{V}$, $\mathbf{U}\times\mathbf{V}$ is perpendicular to both $\mathbf{U}$ and $\mathbf{V}$. Use this fact to show that $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ lies in the plane of the vectors $\mathbf{A}$ and $\mathbf{B}$, so that

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=\lambda\mathbf{A}+\mu\mathbf{B},$$

for some scalars $\lambda$ and $\mu$.

($ii$) To find $\lambda$ and $\mu$, take $\mathbf{i}$ to be the unit vector in the direction of $\mathbf{A}$, and take $\mathbf{j}$ as the unit vector perpendicular to $\mathbf{i}$ in the plane of $\mathbf{A}$ and $\mathbf{B}$. Indicate why it is true that

$$\begin{aligned}\mathbf{A}&=a\mathbf{i}\\\mathbf{B}&=b_1\mathbf{i}+b_2\mathbf{j}.\end{aligned}$$

($iii$) By writing $\mathbf{C}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$, where $\mathbf{k}=\mathbf{i}\times\mathbf{j}$, compute $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ directly and find $\lambda$ and $\mu$. Hence prove

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{B}\cdot\mathbf{C})\mathbf{A}.$$

*Solution.*

($i$) Consider first the case when $\mathbf{A}$ and $\mathbf{B}$ are parallel. This means that the angle $\theta$ between $\mathbf{A}$ and $\mathbf{B}$ is 0 or $\pi$, and so $\sin\theta=0$. So $\mathbf{A}\times\mathbf{B}=\mathbf{0}$ (since $|\mathbf{A}\times\mathbf{B}|=|\mathbf{A}|\,|\mathbf{B}|\sin\theta$). So in this case $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=\mathbf{0}=0\mathbf{A}+0\mathbf{B}$.

Now suppose that $\mathbf{A}$ and $\mathbf{B}$ are not parallel. Fix an origin, $O$, and let $P$ and $Q$ be the points whose position vectors relative to $O$ are $\mathbf{A}$ and $\mathbf{B}$. Then if $\lambda$ and $\mu$ are any scalars, $\lambda\mathbf{A} + \mu\mathbf{B}$ is the position vector of some point in the plane of $O$, $P$ and $Q$. Furthermore, every point in this plane has position vector of the form $\lambda\mathbf{A} + \mu\mathbf{B}$ for some scalars $\lambda$ and $\mu$ (cf. Exercise 12 on p. 22 of the book). Now $\mathbf{A} \times \mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$; so its direction is perpendicular to the plane of $O$, $P$ and $Q$. Thus we can see geometrically that any vector that is perpendicular to $\mathbf{A} \times \mathbf{B}$ must be parallel to something in the plane of $O$, $P$ and $Q$. In particular, since $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ is perpendicular to $\mathbf{A} \times \mathbf{B}$, we have

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \lambda\mathbf{A} + \mu\mathbf{B},$$

for some scalars $\lambda$ and $\mu$.

[Comment: A line $\ell$ that is perpendicular to every line lying in a plane $\mathcal{P}$ is called a *normal* to $\mathcal{P}$. What we have made use of above is that every line perpendicular to the normal $\ell$ is parallel to some line in the plane $\mathcal{P}$.]

(*ii*) The definition of the cross product given at the top of p. 50 of the book is independent of the choice of a coordinate system. So we can choose any coordinate system we like: the values of $\lambda$ and $\mu$ will be unchanged. In particular, we can choose the $x$-axis to be in the direction of $\mathbf{A}$, so that $\mathbf{A} = a\mathbf{i}$ for some real number $a > 0$, and choose the $y$-axis so that the $xy$-plane is the plane of $\mathbf{A}$ and $\mathbf{B}$. Then $\mathbf{B}$ is in the plane of $\mathbf{i}$ and $\mathbf{j}$; so $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$ for some real numbers $b_1$ and $b_2$.

(*iii*) We have $\mathbf{A} \times \mathbf{B} = a\mathbf{i} \times (b_1\mathbf{i} + b_2\mathbf{j}) = ab_2\mathbf{k}$. Hence

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = ab_2\mathbf{k} \times (c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}) = ab_2c_1\mathbf{j} - ab_2c_2\mathbf{i}.$$

On the other hand, we also find that

$$(\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A} = ac_1(b_1\mathbf{i} + b_2\mathbf{j}) - (b_1c_1 + b_2c_2)a\mathbf{i} = ab_2c_1\mathbf{j} - ab_2c_2\mathbf{i},$$

showing that $\lambda = -(\mathbf{B} \cdot \mathbf{C})$ and $\mu = (\mathbf{A} \cdot \mathbf{C})$.

## Answers to Preparatory Questions

**1.** (*i*) $(2, 3, 6)$ and $(4, 4, 7)$. (*ii*) 7 and 9.

(*iii*) For $\overrightarrow{AB}$: 2/7, 3/7 and 6/7. For $\overrightarrow{AC}$: 4/9, 4/9 and 7/9.

**2.** (*i*) $-9$ (*ii*) $-3/\sqrt{14}$ (*iii*) $-5$ (*iv*) $-3$ (*v*) $-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.

**3.** (*i*) $|\mathbf{u}| = |\mathbf{v}| = \sqrt{14}$.

(*ii*) $-4\mathbf{i} - 8\mathbf{j} - 10\mathbf{k}$.

(*iii*) $\sin\theta = \dfrac{6\sqrt{5}}{\sqrt{14}\sqrt{14}} = \dfrac{3\sqrt{5}}{7}$.

(*iv*) This simplifies to $17(\mathbf{u} \times \mathbf{v})$, which equals $17(-4\mathbf{i} - 8\mathbf{j} - 10\mathbf{k})$.

## Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 3.