

Preliminary Reading: Chapter 4 of the Vectors book.

Objectives:

By the end of Week 4, to achieve at least a pass level, you should be able to

4A: perform simple calculations using both scalar and vector products.

4B: use the scalar triple product to calculate the volume of a parallelepiped.

4C: recognise and convert between the parametric (vector and scalar) and the Cartesian forms of the equation of a line.

To achieve higher than a pass level you should be able to

4D: use the vector product to calculate the perpendicular distance from a point to the line through two given points.

4E: calculate the distance between two lines.

4F: use the vector representation of a line to prove theorems in geometry.

Preparatory questions. (Answers are on the next page.)

1. Verify by direct calculation that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w}$ where $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.
2. Find the volume of the parallelepiped having \mathbf{u} , \mathbf{v} and \mathbf{w} of Question 1 as adjacent edges.
3. Given the line ℓ with parametric equation $\mathbf{r} = \mathbf{i} + \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$:
 - (i) find a vector parallel to the line ℓ ;
 - (ii) if $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, express x , y and z in terms of t .

Practice questions

4. Given a point O as origin, orthonormal basis \mathbf{i} , \mathbf{j} , \mathbf{k} and vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{c} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, let A , B and C be the points in space such that $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$.
 - (i) Find a vector perpendicular to the plane containing A , B and C .
 - (ii) Find the perpendicular distance from A to the line through B and C .
 - (iii) Find the area of the triangle ABC .
 - (iv) Show that $(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})$ is perpendicular to the plane of ABC . How is this vector related to the one you found in (i) and to your answer to (iii)?

Solution.

- (i) We have $\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = -2\mathbf{i} + \mathbf{k}$ and $\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = -4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. The vector product $\overrightarrow{BA} \times \overrightarrow{BC}$ is perpendicular to both \overrightarrow{BA} and \overrightarrow{BC} , and therefore perpendicular to all lines lying in the plane through A , B and C .

Applying the formula for the vector product given on p. 52 of the Vectors notes, we find that $\overrightarrow{BA} \times \overrightarrow{BC} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. (You can check that $\overrightarrow{BA} \times \overrightarrow{AC}$ and $\overrightarrow{BC} \times \overrightarrow{AC}$ give the same result. Furthermore, all scalar multiples of $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ are perpendicular to ABC .)

(ii) By a well-known formula (proved on p. 55 of the notes), the area of the triangle ABC is $\frac{1}{2}|\overrightarrow{BA}| |\overrightarrow{BC}| \sin \angle ABC$. This is half the length of $\overrightarrow{BA} \times \overrightarrow{BC}$. But the area also equals $\frac{1}{2}|\overrightarrow{BC}|d$, where d is the perpendicular distance from A to the line through B and C . So $d = \frac{|\overrightarrow{BA} \times \overrightarrow{BC}|}{|\overrightarrow{BC}|} = \frac{\sqrt{1^2+2^2+2^2}}{\sqrt{4^2+1^2+3^2}} = \frac{3}{\sqrt{26}}$.

(iii) The area is $\frac{1}{2}|\overrightarrow{BA} \times \overrightarrow{BC}| = 3/2$.

(iv) We saw in Part (i) that $\overrightarrow{BA} \times \overrightarrow{BC}$ is perpendicular to the plane ABC ; so, effectively, we are asked to prove that $(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})$ is a scalar multiple of $\overrightarrow{BA} \times \overrightarrow{BC}$. Now since $\mathbf{b} \times \mathbf{b} = \mathbf{0}$,

$$\overrightarrow{BA} \times \overrightarrow{BC} = (\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{b}) = \mathbf{a} \times \mathbf{c} - \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{c} = -\mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{a} - \mathbf{b} \times \mathbf{c}$$

as required. Thus $(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})$ is the negative of our answer to Part (i), and its magnitude is twice the area of the triangle ABC .

5. (Class discussion) If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = 0$ is it necessary that $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$?

Solution.

Yes, because $|\mathbf{u}| |\mathbf{v}| \sin \theta = 0$ and $|\mathbf{u}| |\mathbf{v}| \cos \theta = 0$ implies $|\mathbf{u}| |\mathbf{v}| = 0$ and therefore either $|\mathbf{u}| = 0$ or $|\mathbf{v}| = 0$.

6. (Class discussion) Given a vector \mathbf{u} , describe the points whose position vector \mathbf{r} satisfies $\mathbf{r} \cdot (\mathbf{r} - \mathbf{u}) = 0$.

Solution.

Let O be the origin and U be the point whose position vector is \mathbf{u} . Now if \mathbf{r} is the position vector of P , the condition $\mathbf{r} \cdot (\mathbf{r} - \mathbf{u}) = 0$ says that OP is perpendicular to UP . So we are asked to describe all the points P such that OPU is a right-angled triangle with hypotenuse OU . In fact, if M is the midpoint of OU then OPU is right-angled if and only if the distance MP is the same as the (equal) distances OM and UM . (The angle in a semicircle is a right-angle.) The set of all points whose distance from M is specified forms a circle with centre M if we restrict ourselves to a plane; in 3-space we get a sphere with centre M .

In vector terminology, the condition that the distances MP and OM are equal can be written as $|\overrightarrow{MP}| = |\overrightarrow{OM}|$. Now $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OU} = \frac{1}{2}\mathbf{u}$, and $\overrightarrow{MP} = \mathbf{r} - \frac{1}{2}\mathbf{u}$. We show that the condition $|\mathbf{r} - \frac{1}{2}\mathbf{u}| = |\frac{1}{2}\mathbf{u}|$ is indeed the same as the condition $\mathbf{r} \cdot (\mathbf{r} - \mathbf{u}) = 0$. Firstly, $|\mathbf{r} - \frac{1}{2}\mathbf{u}| = |\frac{1}{2}\mathbf{u}|$ if and only if $(\mathbf{r} - \frac{1}{2}\mathbf{u}) \cdot (\mathbf{r} - \frac{1}{2}\mathbf{u}) = (\frac{1}{2}\mathbf{u}) \cdot (\frac{1}{2}\mathbf{u})$. Now since $(\mathbf{r} - \frac{1}{2}\mathbf{u}) \cdot (\mathbf{r} - \frac{1}{2}\mathbf{u}) = \mathbf{r} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{u} + \frac{1}{4}\mathbf{u} \cdot \mathbf{u}$, this equals $(\frac{1}{2}\mathbf{u}) \cdot (\frac{1}{2}\mathbf{u})$ if and only if $\mathbf{r} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{u} = 0$. This is precisely $\mathbf{r} \cdot (\mathbf{r} - \mathbf{u}) = 0$, as claimed.

7. Given the points $A(1, -1, 6)$, $B(2, 1, 0)$, $C(-3, 2, -4)$ and $D(-9, 1, -2)$,

(i) find the equation of the line through A that is parallel to BC , (a) in parametric vector form, and (b) in Cartesian form, and

(ii) show that D lies on this line.

Solution.

(i) $\overrightarrow{OA} = \mathbf{i} - \mathbf{j} + 6\mathbf{k}$ and $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (-3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) - (2\mathbf{i} + \mathbf{j}) = -5\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.
The line through A parallel to \overrightarrow{BC} is

$$(a) \mathbf{r} = \mathbf{i} - \mathbf{j} + 6\mathbf{k} + t(-5\mathbf{i} + \mathbf{j} - 4\mathbf{k}), \quad (b) \frac{x-1}{-5} = \frac{y+1}{1} = \frac{z-6}{-4}.$$

(i) Can we find t such that $-9\mathbf{i} + \mathbf{j} - 2\mathbf{k} = \mathbf{i} - \mathbf{j} + 6\mathbf{k} + t(-5\mathbf{i} + \mathbf{j} - 4\mathbf{k})$? By inspection $t = 2$ satisfies this relation, and so D lies on the line.

Alternatively, since the coordinates of D are $(x, y, z) = (-9, 1, -2)$, we just need to check that these values satisfy the equations in (b). It is correct: $\frac{-9-1}{-5} = \frac{1+1}{1} = \frac{-2-6}{-4} = 2$.

8. (i) Find a vector that is perpendicular to both the line through the points $A(1, -2, -1)$ and $B(4, 0, -3)$ and to the line ℓ whose vector equation is $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} - 6\mathbf{j} - 4\mathbf{k})$.

(ii) Find a point L on AB and a point M on the line ℓ such that \overrightarrow{LM} is perpendicular to both of these lines.

[Hint: Let L be the point on AB such that $\overrightarrow{AL} = \alpha\overrightarrow{AB}$, and M be the point with position vector $\mathbf{m} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \beta(\mathbf{i} - 6\mathbf{j} - 4\mathbf{k})$, where α and β are to be determined by the condition that \overrightarrow{LM} is perpendicular to both lines, and therefore parallel to the vector determined in part (i).]

(iii) Show that the shortest distance between the two lines is $4/3$. [Hint: $|\overrightarrow{LM}|$]. (Can you think of a way to calculate the shortest distance without finding the locations of L and M ?)

Solution.

(i) Since the direction of ℓ is given by $\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$, the vector $\overrightarrow{AB} \times (\mathbf{i} - 6\mathbf{j} - 4\mathbf{k})$ is perpendicular to the line AB and the line ℓ . Now $\overrightarrow{AB} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, and the formula for the cross product gives $\overrightarrow{AB} \times (\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}) = -20\mathbf{i} + 10\mathbf{j} - 20\mathbf{k}$. This vector, or any nonzero scalar multiple of it (such as $-2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ for example) will serve as a correct answer.

(ii) Using the hint, the position vector of L is

$$\overrightarrow{OL} = \overrightarrow{OA} + \overrightarrow{AL} = \overrightarrow{OA} + \alpha\overrightarrow{AB} = \mathbf{a} + \alpha(\mathbf{b} - \mathbf{a}) = \mathbf{i} - 2\mathbf{j} - \mathbf{k} + \alpha(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$$

and M is a point on ℓ with the position vector

$$\overrightarrow{OM} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \beta(\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}).$$

Thus

$$\overrightarrow{LM} = -\overrightarrow{OL} + \overrightarrow{OM} = (\beta - 3\alpha)\mathbf{i} + (4 - 6\beta - 2\alpha)\mathbf{j} + (-4\beta + 2\alpha)\mathbf{k}.$$

For \overrightarrow{LM} to be perpendicular to both lines it must be parallel to $-2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, and hence a scalar multiple of $-2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Thus we need

$$(\beta - 3\alpha)\mathbf{i} + (-6\beta - 2\alpha + 4)\mathbf{j} + (-4\beta + 2\alpha)\mathbf{k} = \gamma(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$

for some scalar γ . Equating the coefficients of \mathbf{i} , \mathbf{j} and \mathbf{k} gives the three equations

$$\begin{aligned}\beta - 3\alpha &= -2\gamma, \\ -6\beta - 2\alpha + 4 &= \gamma, \\ -4\beta + 2\alpha &= -2\gamma.\end{aligned}$$

Adding four times the first equation to the third gives $-10\alpha = -10\gamma$. So $\gamma = \alpha$, and substituting this into the first equation gives $\beta = \alpha$ also. We can now use the second equation to find the common value of the three unknowns, and the result is $\alpha = \beta = \gamma = \frac{4}{9}$, giving

$$\overrightarrow{OL} = (21\mathbf{i} - 10\mathbf{j} - 17\mathbf{k})/9 \quad \text{and} \quad \overrightarrow{OM} = (13\mathbf{i} - 6\mathbf{j} - 25\mathbf{k})/9.$$

(iii) Let \mathcal{P} be the plane through L such that LM is normal to \mathcal{P} . Note that the line AB lies in \mathcal{P} . Similarly, let \mathcal{P}' be the plane through M such that LM is normal to \mathcal{P}' , and note that ℓ lies in \mathcal{P}' . Since \mathcal{P} and \mathcal{P}' are parallel and LM is normal to them both it is clear that LM gives the shortest distance between \mathcal{P} and \mathcal{P}' ; hence certainly the shortest distance between AB and ℓ . Since $\overrightarrow{LM} = \overrightarrow{LO} + \overrightarrow{OM} = \overrightarrow{OM} - \overrightarrow{OL} = \frac{4}{9}(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$, we obtain

$$|\overrightarrow{LM}| = \frac{4}{9}\sqrt{2^2 + 1^2 + 2^2} = \frac{4}{3}.$$

In fact it is not hard to see that the shortest distance between two parallel planes is given by $\overrightarrow{PQ} \cdot \hat{\mathbf{n}}$, where P is any point on one plane, Q any point on the other and $\hat{\mathbf{n}}$ a unit normal to both planes (directed so that the angle between \overrightarrow{PQ} and $\hat{\mathbf{n}}$ is acute). (Construct the line through P normal to the planes and let it meet the other plane at X . Then PX gives the minimal distance between the planes, and by trigonometry in the right-angled triangle PXQ we see that $|\overrightarrow{PX}| = \overrightarrow{PQ} \cos \theta$, where θ is the angle between \overrightarrow{PQ} and $\hat{\mathbf{n}}$.) So in this question the shortest distance equals (for example) $|\overrightarrow{AC} \cdot \hat{\mathbf{n}}|$, where C is the point $(1, 2, -1)$ (which lies on ℓ) and $\hat{\mathbf{n}} = (-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})/3$. Indeed,

$$\overrightarrow{AC} \cdot \hat{\mathbf{n}} = (-4\mathbf{j}) \cdot (-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})/3 = -\frac{4}{3},$$

confirming that the distance is $\frac{4}{3}$. (The angle between \overrightarrow{AC} and $\hat{\mathbf{n}}$ turned out to be obtuse; if we had replaced $\hat{\mathbf{n}}$ by $-\hat{\mathbf{n}}$ the dot product would have come out positive.)

Answers to Preparatory Questions

2. 1

3. (i) $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(ii) $x = 1 + t, y = 2t, z = 1 + 3t$.

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

www.maths.usyd.edu.au/u/UG/JM/MATH1902/

and then do the Web Quiz for Week 4.