

**Preliminary Reading:** Chapter 4 of the Vectors book.

**Objectives:**

By the end of Week 4, to achieve at least a pass level, you should be able to

4A: perform simple calculations using both scalar and vector products.

4B: use the scalar triple product to calculate the volume of a parallelepiped.

4C: recognise and convert between the parametric (vector and scalar) and the Cartesian forms of the equation of a line.

To achieve higher than a pass level you should be able to

4D: use the vector product to calculate the perpendicular distance from a point to the line through two given points.

4E: calculate the distance between two lines.

4F: use the vector representation of a line to prove theorems in geometry.

**Preparatory questions.** (Answers are on the next page.)

1. Verify by direct calculation that  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w}$  where  $\mathbf{u} = \mathbf{i} + \mathbf{j}$ ,  $\mathbf{v} = \mathbf{j} + 2\mathbf{k}$  and  $\mathbf{w} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ .
2. Find the volume of the parallelepiped having  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  of Question 1 as adjacent edges.
3. Given the line  $\ell$  with parametric equation  $\mathbf{r} = \mathbf{i} + \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ :
  - (i) find a vector parallel to the line  $\ell$ ;
  - (ii) if  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ , express  $x$ ,  $y$  and  $z$  in terms of  $t$ .

**Practice questions**

4. Given a point  $O$  as origin, orthonormal basis  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  and vectors  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ ,  $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j}$  and  $\mathbf{c} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ , let  $A$ ,  $B$  and  $C$  be the points in space such that  $\mathbf{a} = \overrightarrow{OA}$ ,  $\mathbf{b} = \overrightarrow{OB}$  and  $\mathbf{c} = \overrightarrow{OC}$ .
  - (i) Find a vector perpendicular to the plane containing  $A$ ,  $B$  and  $C$ .
  - (ii) Find the perpendicular distance from  $A$  to the line through  $B$  and  $C$ .
  - (iii) Find the area of the triangle  $ABC$ .
  - (iv) Show that  $(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})$  is perpendicular to the plane of  $ABC$ . How is this vector related to the one you found in (i) and to your answer to (iii)?

*Solution.*

- (i) We have  $\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = -2\mathbf{i} + \mathbf{k}$  and  $\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = -4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ . The vector product  $\overrightarrow{BA} \times \overrightarrow{BC}$  is perpendicular to both  $\overrightarrow{BA}$  and  $\overrightarrow{BC}$ , and therefore perpendicular to all lines lying in the plane through  $A$ ,  $B$  and  $C$ .

Applying the formula for the vector product given on p. 52 of the Vectors notes, we find that  $\overrightarrow{BA} \times \overrightarrow{BC} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ . (You can check that  $\overrightarrow{BA} \times \overrightarrow{AC}$  and  $\overrightarrow{BC} \times \overrightarrow{AC}$  give the same result. Furthermore, all scalar multiples of  $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$  are perpendicular to  $ABC$ .)

(ii) By a well-known formula (proved on p. 55 of the notes), the area of the triangle  $ABC$  is  $\frac{1}{2}|\overrightarrow{BA}| |\overrightarrow{BC}| \sin \angle ABC$ . This is half the length of  $\overrightarrow{BA} \times \overrightarrow{BC}$ . But the area also equals  $\frac{1}{2}|\overrightarrow{BC}|d$ , where  $d$  is the perpendicular distance from  $A$  to the line through  $B$  and  $C$ . So  $d = \frac{|\overrightarrow{BA} \times \overrightarrow{BC}|}{|\overrightarrow{BC}|} = \frac{\sqrt{1^2+2^2+2^2}}{\sqrt{4^2+1^2+3^2}} = \frac{3}{\sqrt{26}}$ .

(iii) The area is  $\frac{1}{2}|\overrightarrow{BA} \times \overrightarrow{BC}| = 3/2$ .

(iv) We saw in Part (i) that  $\overrightarrow{BA} \times \overrightarrow{BC}$  is perpendicular to the plane  $ABC$ ; so, effectively, we are asked to prove that  $(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})$  is a scalar multiple of  $\overrightarrow{BA} \times \overrightarrow{BC}$ . Now since  $\mathbf{b} \times \mathbf{b} = \mathbf{0}$ ,

$$\overrightarrow{BA} \times \overrightarrow{BC} = (\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{b}) = \mathbf{a} \times \mathbf{c} - \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{c} = -\mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{a} - \mathbf{b} \times \mathbf{c}$$

as required. Thus  $(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})$  is the negative of our answer to Part (i), and its magnitude is twice the area of the triangle  $ABC$ .

5. (Class discussion) If  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$  and  $\mathbf{u} \cdot \mathbf{v} = 0$  is it necessary that  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{v} = \mathbf{0}$ ?

*Solution.*

Yes, because  $|\mathbf{u}| |\mathbf{v}| \sin \theta = 0$  and  $|\mathbf{u}| |\mathbf{v}| \cos \theta = 0$  implies  $|\mathbf{u}| |\mathbf{v}| = 0$  and therefore either  $|\mathbf{u}| = 0$  or  $|\mathbf{v}| = 0$ .

6. (Class discussion) Given a vector  $\mathbf{u}$ , describe the points whose position vector  $\mathbf{r}$  satisfies  $\mathbf{r} \cdot (\mathbf{r} - \mathbf{u}) = 0$ .

*Solution.*

Let  $O$  be the origin and  $U$  be the point whose position vector is  $\mathbf{u}$ . Now if  $\mathbf{r}$  is the position vector of  $P$ , the condition  $\mathbf{r} \cdot (\mathbf{r} - \mathbf{u}) = 0$  says that  $OP$  is perpendicular to  $UP$ . So we are asked to describe all the points  $P$  such that  $OPU$  is a right-angled triangle with hypotenuse  $OU$ . In fact, if  $M$  is the midpoint of  $OU$  then  $OPU$  is right-angled if and only if the distance  $MP$  is the same as the (equal) distances  $OM$  and  $UM$ . (The angle in a semicircle is a right-angle.) The set of all points whose distance from  $M$  is specified forms a circle with centre  $M$  if we restrict ourselves to a plane; in 3-space we get a sphere with centre  $M$ .

In vector terminology, the condition that the distances  $MP$  and  $OM$  are equal can be written as  $|\overrightarrow{MP}| = |\overrightarrow{OM}|$ . Now  $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OU} = \frac{1}{2}\mathbf{u}$ , and  $\overrightarrow{MP} = \mathbf{r} - \frac{1}{2}\mathbf{u}$ . We show that the condition  $|\mathbf{r} - \frac{1}{2}\mathbf{u}| = |\frac{1}{2}\mathbf{u}|$  is indeed the same as the condition  $\mathbf{r} \cdot (\mathbf{r} - \mathbf{u}) = 0$ . Firstly,  $|\mathbf{r} - \frac{1}{2}\mathbf{u}| = |\frac{1}{2}\mathbf{u}|$  if and only if  $(\mathbf{r} - \frac{1}{2}\mathbf{u}) \cdot (\mathbf{r} - \frac{1}{2}\mathbf{u}) = (\frac{1}{2}\mathbf{u}) \cdot (\frac{1}{2}\mathbf{u})$ . Now since  $(\mathbf{r} - \frac{1}{2}\mathbf{u}) \cdot (\mathbf{r} - \frac{1}{2}\mathbf{u}) = \mathbf{r} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{u} + \frac{1}{4}\mathbf{u} \cdot \mathbf{u}$ , this equals  $(\frac{1}{2}\mathbf{u}) \cdot (\frac{1}{2}\mathbf{u})$  if and only if  $\mathbf{r} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{u} = 0$ . This is precisely  $\mathbf{r} \cdot (\mathbf{r} - \mathbf{u}) = 0$ , as claimed.

7. Given the points  $A(1, -1, 6)$ ,  $B(2, 1, 0)$ ,  $C(-3, 2, -4)$  and  $D(-9, 1, -2)$ ,

(i) find the equation of the line through  $A$  that is parallel to  $BC$ , (a) in parametric vector form, and (b) in Cartesian form, and

(ii) show that  $D$  lies on this line.

*Solution.*

(i)  $\overrightarrow{OA} = \mathbf{i} - \mathbf{j} + 6\mathbf{k}$  and  $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (-3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}) - (2\mathbf{i} + \mathbf{j}) = -5\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ .  
The line through  $A$  parallel to  $\overrightarrow{BC}$  is

$$(a) \mathbf{r} = \mathbf{i} - \mathbf{j} + 6\mathbf{k} + t(-5\mathbf{i} + \mathbf{j} - 4\mathbf{k}), \quad (b) \frac{x-1}{-5} = \frac{y+1}{1} = \frac{z-6}{-4}.$$

(i) Can we find  $t$  such that  $-9\mathbf{i} + \mathbf{j} - 2\mathbf{k} = \mathbf{i} - \mathbf{j} + 6\mathbf{k} + t(-5\mathbf{i} + \mathbf{j} - 4\mathbf{k})$ ? By inspection  $t = 2$  satisfies this relation, and so  $D$  lies on the line.

Alternatively, since the coordinates of  $D$  are  $(x, y, z) = (-9, 1, -2)$ , we just need to check that these values satisfy the equations in (b). It is correct:  $\frac{-9-1}{-5} = \frac{1+1}{1} = \frac{-2-6}{-4} = 2$ .

8. (i) Find a vector that is perpendicular to both the line through the points  $A(1, -2, -1)$  and  $B(4, 0, -3)$  and to the line  $\ell$  whose vector equation is  $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} - 6\mathbf{j} - 4\mathbf{k})$ .

(ii) Find a point  $L$  on  $AB$  and a point  $M$  on the line  $\ell$  such that  $\overrightarrow{LM}$  is perpendicular to both of these lines.

[Hint: Let  $L$  be the point on  $AB$  such that  $\overrightarrow{AL} = \alpha\overrightarrow{AB}$ , and  $M$  be the point with position vector  $\mathbf{m} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \beta(\mathbf{i} - 6\mathbf{j} - 4\mathbf{k})$ , where  $\alpha$  and  $\beta$  are to be determined by the condition that  $\overrightarrow{LM}$  is perpendicular to both lines, and therefore parallel to the vector determined in part (i).]

(iii) Show that the shortest distance between the two lines is  $4/3$ . [Hint:  $|\overrightarrow{LM}|$ ]. (Can you think of a way to calculate the shortest distance without finding the locations of  $L$  and  $M$ ?)

*Solution.*

(i) Since the direction of  $\ell$  is given by  $\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$ , the vector  $\overrightarrow{AB} \times (\mathbf{i} - 6\mathbf{j} - 4\mathbf{k})$  is perpendicular to the line  $AB$  and the line  $\ell$ . Now  $\overrightarrow{AB} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ , and the formula for the cross product gives  $\overrightarrow{AB} \times (\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}) = -20\mathbf{i} + 10\mathbf{j} - 20\mathbf{k}$ . This vector, or any nonzero scalar multiple of it (such as  $-2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$  for example) will serve as a correct answer.

(ii) Using the hint, the position vector of  $L$  is

$$\overrightarrow{OL} = \overrightarrow{OA} + \overrightarrow{AL} = \overrightarrow{OA} + \alpha\overrightarrow{AB} = \mathbf{a} + \alpha(\mathbf{b} - \mathbf{a}) = \mathbf{i} - 2\mathbf{j} - \mathbf{k} + \alpha(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$$

and  $M$  is a point on  $\ell$  with the position vector

$$\overrightarrow{OM} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \beta(\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}).$$

Thus

$$\overrightarrow{LM} = -\overrightarrow{OL} + \overrightarrow{OM} = (\beta - 3\alpha)\mathbf{i} + (4 - 6\beta - 2\alpha)\mathbf{j} + (-4\beta + 2\alpha)\mathbf{k}.$$

For  $\overrightarrow{LM}$  to be perpendicular to both lines it must be parallel to  $-2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ , and hence a scalar multiple of  $-2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ . Thus we need

$$(\beta - 3\alpha)\mathbf{i} + (-6\beta - 2\alpha + 4)\mathbf{j} + (-4\beta + 2\alpha)\mathbf{k} = \gamma(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$

for some scalar  $\gamma$ . Equating the coefficients of  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  gives the three equations

$$\begin{aligned}\beta - 3\alpha &= -2\gamma, \\ -6\beta - 2\alpha + 4 &= \gamma, \\ -4\beta + 2\alpha &= -2\gamma.\end{aligned}$$

Adding four times the first equation to the third gives  $-10\alpha = -10\gamma$ . So  $\gamma = \alpha$ , and substituting this into the first equation gives  $\beta = \alpha$  also. We can now use the second equation to find the common value of the three unknowns, and the result is  $\alpha = \beta = \gamma = \frac{4}{9}$ , giving

$$\overrightarrow{OL} = (21\mathbf{i} - 10\mathbf{j} - 17\mathbf{k})/9 \quad \text{and} \quad \overrightarrow{OM} = (13\mathbf{i} - 6\mathbf{j} - 25\mathbf{k})/9.$$

(iii) Let  $\mathcal{P}$  be the plane through  $L$  such that  $LM$  is normal to  $\mathcal{P}$ . Note that the line  $AB$  lies in  $\mathcal{P}$ . Similarly, let  $\mathcal{P}'$  be the plane through  $M$  such that  $LM$  is normal to  $\mathcal{P}'$ , and note that  $\ell$  lies in  $\mathcal{P}'$ . Since  $\mathcal{P}$  and  $\mathcal{P}'$  are parallel and  $LM$  is normal to them both it is clear that  $LM$  gives the shortest distance between  $\mathcal{P}$  and  $\mathcal{P}'$ ; hence certainly the shortest distance between  $AB$  and  $\ell$ . Since  $\overrightarrow{LM} = \overrightarrow{LO} + \overrightarrow{OM} = \overrightarrow{OM} - \overrightarrow{OL} = \frac{4}{9}(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ , we obtain

$$|\overrightarrow{LM}| = \frac{4}{9}\sqrt{2^2 + 1^2 + 2^2} = \frac{4}{3}.$$

In fact it is not hard to see that the shortest distance between two parallel planes is given by  $\overrightarrow{PQ} \cdot \hat{\mathbf{n}}$ , where  $P$  is any point on one plane,  $Q$  any point on the other and  $\hat{\mathbf{n}}$  a unit normal to both planes (directed so that the angle between  $\overrightarrow{PQ}$  and  $\hat{\mathbf{n}}$  is acute). (Construct the line through  $P$  normal to the planes and let it meet the other plane at  $X$ . Then  $PX$  gives the minimal distance between the planes, and by trigonometry in the right-angled triangle  $PXQ$  we see that  $|\overrightarrow{PX}| = \overrightarrow{PQ} \cos \theta$ , where  $\theta$  is the angle between  $\overrightarrow{PQ}$  and  $\hat{\mathbf{n}}$ .) So in this question the shortest distance equals (for example)  $|\overrightarrow{AC} \cdot \hat{\mathbf{n}}|$ , where  $C$  is the point  $(1, 2, -1)$  (which lies on  $\ell$ ) and  $\hat{\mathbf{n}} = (-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})/3$ . Indeed,

$$\overrightarrow{AC} \cdot \hat{\mathbf{n}} = (-4\mathbf{j}) \cdot (-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})/3 = -\frac{4}{3},$$

confirming that the distance is  $\frac{4}{3}$ . (The angle between  $\overrightarrow{AC}$  and  $\hat{\mathbf{n}}$  turned out to be obtuse; if we had replaced  $\hat{\mathbf{n}}$  by  $-\hat{\mathbf{n}}$  the dot product would have come out positive.)

### Answers to Preparatory Questions

2. 1

3. (i)  $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ .

(ii)  $x = 1 + t, y = 2t, z = 1 + 3t$ .

### Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

[www.maths.usyd.edu.au/u/UG/JM/MATH1902/](http://www.maths.usyd.edu.au/u/UG/JM/MATH1902/)

and then do the Web Quiz for Week 4.