THE UNIVERSITY OF SYDNEY

MATH1902 Linear Algebra (advanced)

Exercises for Week 5 (beginning March 30) Semester 1 2009

Preliminary Reading:

Chapter 4 of the Vectors book.

Objectives:

By the end of Week 5, to achieve at least a pass level, you should be able to

5A: find the equation of the line through two given points,

5B: find the equation of the plane through a given point and perpendicular to a given vector,

5C: find the equation of the plane through three given non-collinear points.

To achieve higher than a pass level you should be able to

5D: use vector methods, including scalar and vector products to solve problems in geometry involving lines, planes, and other geometric figures.

Preparatory questions. (Answers are on the next page.)

1. Find the equation of the line through the points $A\,(1, 2, 3)$ and $B\,(-3, 0, 4)$ in both vector and Cartesian form.

2. Determine whether the points $M\,(1, 1, 1)$ and $N\,(5, 4, 2)$ are on the line through A and B above.

3. Find the Cartesian equation of the plane through the point $(1, 2, 0)$ and perpendicular to $2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$.

Practice questions

4. Find, in Cartesian form, the equation of the plane which passes through the point $A\,(2, -1, 1)$ and which is perpendicular to the vector from the origin to $B\,(-1, 1, 3)$.

Solution.

The position vector of a point in the plane is $\mathbf{a} = \vec{OA} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and a vector perpendicular to the plane is $\mathbf{b} = \vec{OB} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. The equation of the plane is thus $(\mathbf{r} - 2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = 0$ or $-x + y + 3z = 0$.

5. Find the equation of the plane containing the points $P\,(2, 1, -3)$, $Q\,(4, -1, 2)$ and $R\,(3, 0, 1)$.

Solution.

The vectors $\vec{PQ} = 2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $\vec{PR} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$ are parallel to the plane and so their vector product is perpendicular to the plane. For the vector product we have

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 5 \\ 1 & -1 & 4 \end{vmatrix} = -3\mathbf{i} - 3\mathbf{j}.$$

and so $\mathbf{i}+\mathbf{j}$ is perpendicular to the plane. Thus the required equation is

$$(\mathbf{r}-(2\mathbf{i}+\mathbf{j}-3\mathbf{k}))\cdot(\mathbf{i}+\mathbf{j})=0.$$

That is, $x+y=3$ and it is easy to check that the coordinates of the given points satisfy this equation.

6. Find the intersection of the line joining the points $A\,(1,-2,-1)$ and $B\,(2,3,1)$ with the plane passing through the points $P\,(2,1,-3)$, $Q\,(4,-1,2)$ and $R\,(3,0,1)$.

Solution.

The equation of the line is

$$\mathbf{r}=(\mathbf{i}-2\mathbf{j}-\mathbf{k})+t(\mathbf{i}+5\mathbf{j}+2\mathbf{k})$$

which in coordinate form is

$$x=1+t \quad y=-2+5t \quad z=-1+2t.$$

The equation of the plane is $x+y=3$, as found in the previous question. Thus the line meets the plane where $(1+t)+(-2+5t)=3$. That is, $t=2/3$ and so the point of intersection is $(5/3,4/3,1/3)$.

7. (*i*) Show that the vector equation of the plane passing through the three points $A\,(-1,4,2)$, $B\,(2,4,1)$ and $C\,(-1,0,1)$ is

$$(\mathbf{r}+\mathbf{i}-\mathbf{k})\cdot(-4\mathbf{i}+3\mathbf{j}-12\mathbf{k})=0.$$

(*ii*) Hence find the shortest distance from the point $D\,(1,-2,1)$ to the plane. [Hint: Consider the component of $\vec{DA}$ in the direction of the normal to the plane. Ans: 14/13].

Solution.

(*i*) A vector perpendicular to the plane is given by $\vec{AB}\times\vec{BC}=(3\mathbf{i}-\mathbf{k})\times(-3\mathbf{i}-4\mathbf{j})=(-4\mathbf{i}+3\mathbf{j}-12\mathbf{k})$. The vector equation of the plane is $(\mathbf{r}-\mathbf{r}_0)\cdot(-4\mathbf{i}+3\mathbf{j}-12\mathbf{k})=0$, where $\mathbf{r}_0$ is the position vector of any point in the plane. If we take $\mathbf{r}_0$ to be $\vec{OC}$ we obtain the stated result. (The corresponding Cartesian equation is $4x-3y+12z=8$. Note that the same equation would result if we chose $\mathbf{r}_0=\vec{OA}$ or $\vec{OB}$.)

(*ii*) The shortest distance to the plane is the absolute value of the component of $\vec{DA}$ in the direction normal to the plane. This direction may be taken to be given by $-4\mathbf{i}+3\mathbf{j}-12\mathbf{k}$. Now $\vec{DA}=-2\mathbf{i}+6\mathbf{j}+\mathbf{k}$ so required component is

$$\begin{aligned}&(-2\mathbf{i}+6\mathbf{j}+\mathbf{k})\cdot(-4\mathbf{i}+3\mathbf{j}-12\mathbf{k})/\sqrt{(-4)^2+3^2+(-12)^2}\\&=(8+18-12)/13\\&=14/13.\end{aligned}$$

8. The point P with position vector $\mathbf{r}=\vec{OP}$ lies on the sphere with centre C and radius a if and only if $|\mathbf{r}-\mathbf{c}|=a$, where $\mathbf{c}=\vec{OC}$. (Thus $|\mathbf{r}-\mathbf{c}|=a$ is the equation of the sphere.)

(i) Show that the equation of the sphere can be written in the alternative form $|\mathbf{r}|^2 - 2\mathbf{c}\cdot\mathbf{r} + |\mathbf{c}|^2 - a^2 = 0$.

(ii) Given a point B on the sphere with position vector $\mathbf{b} = \vec{OB}$ show that the line $\mathbf{r} = \mathbf{b} + t\mathbf{d}$ through B in the direction $\mathbf{d}$ meets the sphere where $t = 0$ and where $t = -2\mathbf{d}\cdot(\mathbf{b}-\mathbf{c})/|\mathbf{d}|^2$.

(iii) Deduce from the previous part of the question that the line is tangent to the sphere if and only if $\mathbf{d}\cdot(\mathbf{b}-\mathbf{c}) = 0$.

(iv) What is the vector equation of the tangent plane to the sphere at B?

Solution.

(i) Squaring $|\mathbf{r}-\mathbf{c}| = a$ gives $a^2 = |\mathbf{r}-\mathbf{c}|^2 = (\mathbf{r}-\mathbf{c})\cdot(\mathbf{r}-\mathbf{c}) = |\mathbf{r}|^2 - 2\mathbf{c}\cdot\mathbf{r} + |\mathbf{c}|^2$.

(ii) Substituting $\mathbf{r} = \mathbf{b} + t\mathbf{d}$ into $|\mathbf{r}-\mathbf{c}| = a$ (or the previous equation) yields the quadratic equation in t:

$$|\mathbf{b}-\mathbf{c}|^2 + 2t\mathbf{d}\cdot(\mathbf{b}-\mathbf{c}) + t^2|\mathbf{d}|^2 = a^2.$$

But B is on the sphere and therefore $|\mathbf{b}-\mathbf{c}|^2 = a^2$. Thus the quadratic simplifies to $t(|\mathbf{d}|^2 t + 2\mathbf{d}\cdot(\mathbf{b}-\mathbf{c}) = 0$ and its solutions are $t = 0$ and $t = -2\mathbf{d}\cdot(\mathbf{b}-\mathbf{c})/|\mathbf{d}|^2$.

(iii) The line becomes tangent to the sphere when the two points of intersection coincide. That is, when $\mathbf{d}\cdot(\mathbf{b}-\mathbf{c}) = 0$; i.e., the tangent is perpendicular to $\mathbf{b}-\mathbf{c}$.

(iv) The equation of the tangent plane at B is $(\mathbf{r}-\mathbf{b})\cdot(\mathbf{b}-\mathbf{c}) = 0$.

Answers to Preparatory Questions

1. The vector parametric equation of the line is $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(-4\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ and the Cartesian form is

$$\frac{x-1}{-4} = \frac{y-2}{-2} = \frac{z-3}{1}.$$

2. Substituting the coordinates of M and N into the previous equations shows that N is on the line but M is not.

3. $2x - 5y + z = -8$.

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 5.