

Preliminary Reading:

Chapter 1 of the Linear Algebra book.

Objectives:

By the end of Week 6, to achieve at least a pass level, you should be able to

6A: solve equations using Gaussian elimination and elementary row operations,

6B: explain why the row operations you use to solve equations must be reversible.

To achieve higher than a pass level you should be able to

6C: understand the connection between solving simultaneous linear equations and finding the intersection of lines and planes in space.

Preparatory questions. (Answers are on the next page.)

1. For the following pair of linear equations, write down an elementary row operation which eliminates x from the second equation:
- $$\begin{aligned} 4x - 2y + z &= 3 \\ -7x + 2y - 3z &= 2 \end{aligned}$$

2. Write down the *augmented matrix* for each of the following systems of linear equations:

$$\begin{array}{ll} (i) & \begin{aligned} x + 2y + z &= 7 \\ 4x - y + 8z &= 3 \\ -x + 3y - 3z &= 2 \end{aligned} & (ii) & \begin{aligned} -x + y + z &= 0 \\ 11x - 2y + 8z &= 0 \\ x + y - z &= 0 \end{aligned} \end{array}$$

3. Given a system of linear equations, is the operation of *simultaneously* applying the row operations $R_2 := R_2 - R_1$ and $R_1 := R_1 - R_2$ reversible? If not, why not?

Practice questions

4. Solve the following systems of linear equations by forming the augmented coefficient matrix and performing elementary row operations.

$$\begin{array}{ll} (i) & \begin{aligned} x + y + 2z &= 8 \\ -x - 2y + 3z &= 1 \\ 3x - 7y + 4z &= 10 \end{aligned} & (ii) & \begin{aligned} 2x + 2y + 2z &= 0 \\ -2x + 5y + 2z &= 0 \\ -7x + 7y + z &= 0 \end{aligned} \\ (iii) & \begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x &\quad - 3w = -3 \end{aligned} & (iv) & \begin{aligned} 2x - 3y &= -2 \\ 2x + y &= 1 \\ 3x + 2y &= 1 \end{aligned} \end{array}$$

Solution.

(i) Performing elementary row operations we obtain

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] &\xrightarrow{\substack{R_2:=R_2+R_1 \\ R_3:=R_3-3R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right] &\xrightarrow{\substack{R_2:=-R_2 \\ R_3:=-R_3}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 10 & 2 & 14 \end{array} \right] \\ &\xrightarrow{R_3:=R_3-10R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 52 & 104 \end{array} \right] &\xrightarrow{R_3:=(1/52)R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right]. \end{aligned}$$

The last matrix describes the following system

$$\begin{aligned} x + y + 2z &= 8 \\ y - 5z &= -9 \\ z &= 2, \end{aligned}$$

and, substituting back, $z = 2$ gives $y = 5z - 9 = 1$ and $x = -y - 2z + 8 = 3$.

(ii)

$$\begin{aligned} \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 0 \\ -7 & 7 & 1 & 0 \end{array} \right] &\xrightarrow{R_1:=(1/2)R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 0 \\ -7 & 7 & 1 & 0 \end{array} \right] &\xrightarrow{\substack{R_2:=R_2+2R_1 \\ R_3:=R_3+7R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 0 \\ 0 & 14 & 8 & 0 \end{array} \right] \\ &\xrightarrow{R_2:=(1/7)R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & 0 \\ 0 & 14 & 8 & 0 \end{array} \right] &\xrightarrow{\substack{R_3:=R_3-14R_2 \\ R_1:=R_1-R_2}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{7} & 0 \\ 0 & 1 & \frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

The leading variables are x and y , and z is free. Setting z equal to an arbitrary parameter t and solving using back substitution gives $z = t$, $y = -\frac{4}{7}t$ and $x = -\frac{3}{7}t$. Thus the general solution is $(x, y, z) = (-\frac{3}{7}t, -\frac{4}{7}t, t)$, where t is arbitrary.

(iii)

$$\begin{aligned} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array} \right] &\xrightarrow{\substack{R_2:=R_2-2R_1 \\ R_3:=R_3+R_1 \\ R_4:=R_4-3R_1}} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array} \right] \\ &\xrightarrow{R_2:=(1/3)R_2} \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array} \right] &\xrightarrow{\substack{R_3:=R_3-R_2 \\ R_4:=R_4-3R_2 \\ R_1:=R_1+R_2}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

The leading entries of the two nonzero rows occur in the x and y columns. So x, y are the leading variables, w, z the free variables. Set $w = t_1$ and $z = t_2$ (arbitrary parameters). Back substitution gives $y = 2t_2$ and $x = t_1 - 1$. So the general solution is $(x, y, z, w) = (t_1 - 1, 2t_2, t_2, t_1)$, where t_1 and t_2 are arbitrary.

(iv)

$$\begin{aligned} \left[\begin{array}{cc|c} 2 & -3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{array} \right] &\xrightarrow{R_1:=(1/2)R_1} \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{array} \right] &\xrightarrow{\substack{R_2:=R_2-2R_1 \\ R_3:=R_3-3R_1}} \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & -1 \\ 0 & 4 & 3 \\ 0 & \frac{13}{2} & 4 \end{array} \right] \\ &\xrightarrow{R_2:=(1/4)R_2} \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & -1 \\ 0 & 1 & \frac{3}{4} \\ 0 & \frac{13}{2} & 4 \end{array} \right] &\xrightarrow{\substack{R_3:=R_3-\frac{13}{2}R_2 \\ R_1:=R_1+\frac{3}{2}R_2}} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{8} \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & -\frac{7}{8} \end{array} \right] \end{aligned}$$

The last matrix represents an inconsistent system. Thus the original equations have no solution.

5. (i) Show that the lines in three dimensional space whose equations are

$$\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$$

and

$$\frac{x-8}{7} = y-4 = \frac{z-5}{3}$$

have a point of intersection, and find it. (Do this either by solving a system of four equations in x , y and z or by using parametric equations.)

- (ii) Find the equation of the plane containing the above two lines.

Solution.

- (i) If a point (x, y, z) lies on both lines then it satisfies the four equations

$$\frac{x-5}{4} = \frac{y-7}{4}, \quad \frac{y-7}{4} = \frac{z+3}{-5}, \quad \frac{x-8}{7} = y-4, \quad y-4 = \frac{z-5}{3}.$$

There is such a point if the equations are consistent.

Let us bring the system to row echelon form. Clearing fractions the equations become

$$\begin{aligned} x - y &= -2 \\ -5y - 4z &= -23 \\ x - 7y &= -20 \\ 3y - z &= 7. \end{aligned}$$

Applying row operations to the augmented matrix gives

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & -5 & -4 & -23 \\ 1 & -7 & 0 & -20 \\ 0 & 3 & -1 & 7 \end{array} \right] & \xrightarrow{R_3 := R_3 - R_1} & \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & -5 & -4 & -23 \\ 0 & -6 & 0 & -18 \\ 0 & 3 & -1 & 7 \end{array} \right] & \xrightarrow{R_2 \leftrightarrow R_3} & \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & -6 & 0 & -18 \\ 0 & -5 & -4 & -23 \\ 0 & 3 & -1 & 7 \end{array} \right] \\ & \xrightarrow{R_2 := (1/6)R_2} & \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & -5 & -4 & -23 \\ 0 & 3 & -1 & 7 \end{array} \right] & \xrightarrow{\begin{array}{l} R_3 := R_3 + 5R_2 \\ R_4 := R_4 - 3R_2 \\ R_1 := R_1 + R_2 \end{array}} & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & -1 & -2 \end{array} \right] \\ & \xrightarrow{\begin{array}{l} R_3 := (-1/4)R_3 \\ R_4 := -R_4 \end{array}} & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array} \right] & \xrightarrow{R_4 := R_4 - R_3} & \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

So the equations are consistent, with unique solution $(x, y, z) = (1, 3, 2)$.

Alternatively, the parametric vector forms of the equations of the two lines are

$$\begin{aligned} \mathbf{r} &= 5\mathbf{i} + 7\mathbf{j} - 3\mathbf{k} + t(4\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \\ \mathbf{r} &= 8\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} + s(7\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \end{aligned}$$

where the parameter t determines a point on the first line and s similarly determines a point on the second line. There is a point lying on both lines if and only if values of t and s can be found such that

$$5\mathbf{i} + 7\mathbf{j} - 3\mathbf{k} + t(4\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) = 8\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} + s(7\mathbf{i} + \mathbf{j} + 3\mathbf{k}).$$

Equating the coefficients of \mathbf{i} , \mathbf{j} and \mathbf{k} , we see that the question is whether the system

$$\begin{aligned} 5 + 4t &= 8 + 7s \\ 7 + 4t &= 4 + s \\ -3 - 5t &= 5 + 3s \end{aligned}$$

is consistent. So this method gives us a system of three equations in two unknowns, whereas the previous method gave four equations in three unknowns. Performing row operations on the augmented matrix gives

$$\left[\begin{array}{cc|c} 4 & -7 & 3 \\ 4 & -1 & -3 \\ -5 & -3 & 8 \end{array} \right] \xrightarrow{R_1 := (1/4)R_1} \left[\begin{array}{cc|c} 1 & -7/4 & 3/4 \\ 4 & -1 & -3 \\ -5 & -3 & 8 \end{array} \right] \xrightarrow{\substack{R_2 := R_2 - 4R_1 \\ R_3 := R_3 + 5R_1}} \left[\begin{array}{cc|c} 1 & -7/4 & 3/4 \\ 0 & 6 & -6 \\ 0 & -47/4 & 47/4 \end{array} \right],$$

and it is clear that the system is consistent. (Indeed, the solution is $t = s = -1$.)

(ii) The first line passes through $(5, 7, -3)$ and is parallel to $4\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$, while the second line passes through $(8, 4, 5)$ and is parallel to $7\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. If \mathcal{P} is a plane containing both of these lines, then a normal to \mathcal{P} is given by the cross product

$$(4\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \times (7\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = 17\mathbf{i} - 47\mathbf{j} - 24\mathbf{k}.$$

Since \mathcal{P} is the plane through $(1, 3, 2)$ with this normal, the equation of \mathcal{P} is

$$(17\mathbf{i} - 47\mathbf{j} - 24\mathbf{k}) \cdot ((x - 1)\mathbf{i} + (y - 3)\mathbf{j} + (z - 2)\mathbf{k}) = 0,$$

which simplifies to $17x - 47y - 24z = -172$. (This may be checked by showing that the three points $(5, 7, -3)$, $(8, 4, 5)$ and $(1, 3, 2)$ all lie on the plane with this equation.)

Alternatively, since $(1, 3, 2)$ lies on both lines and the vectors $4\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $7\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ give directions of two non-parallel lines in the plane, the parametric vector form of the equation of the plane is

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k} + t(4\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) + s(7\mathbf{i} + \mathbf{j} + 3\mathbf{k}),$$

and this is equivalent to the linear system

$$\begin{aligned} 4t + 7s &= x - 1 \\ 4t + s &= y - 3 \\ -5t + 3s &= z - 2 \end{aligned}$$

Doing a sequence of row operations very similar to ones that were done above leads to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & \frac{7}{4} & \frac{1}{4}x - \frac{1}{4} \\ 0 & -6 & y - x - 2 \\ 0 & \frac{47}{4} & \frac{5}{4}x + z - \frac{13}{4} \end{array} \right].$$

The next step is to divide the second row by -6 , and it perhaps helps to multiply the last row by 4 to clear some fractions. We obtain

$$\left[\begin{array}{ccc|c} 1 & \frac{7}{4} & & \\ 0 & 1 & -\frac{1}{6}y + \frac{1}{6}x + \frac{1}{3} & \\ 0 & 47 & 5x + 4z - 13 & \end{array} \right] \xrightarrow{R_3 := R_3 - 47R_2} \left[\begin{array}{ccc|c} 1 & \frac{7}{4} & & \\ 0 & 1 & -\frac{1}{6}y + \frac{1}{6}x + \frac{1}{3} & \\ 0 & 0 & -\frac{17}{6}x + \frac{47}{6}y + 4z - \frac{86}{3} & \end{array} \right]$$

and see that the system is consistent if and only if $-17x + 47y + 24z - 172 = 0$, which is the same as the equation we obtained above.

6. Show that if t , u and v are distinct numbers then the following simultaneous linear equations in the unknowns a , b and c have a unique solution, for any values of the constants p , q and r :

$$\begin{aligned} a + tb + t^2c &= p \\ a + ub + u^2c &= q \\ a + vb + v^2c &= r. \end{aligned}$$

Hence show that there is a graph $y = a + bx + cx^2$ passing through the points (t, p) , (u, q) and (v, r) .

Solution.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & t & t^2 & p \\ 1 & u & u^2 & q \\ 1 & v & v^2 & r \end{array} \right].$$

If by applying row operations to this we can obtain an augmented matrix that has the form

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array} \right]$$

(where there can be any values in the positions marked by asterisks) then the system will have a unique solution, because the last equation will determine z uniquely, then the second equation will determine y uniquely, and then the first equation will determine x uniquely. We shall show that in fact this occurs.

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & t & t^2 & p \\ 1 & u & u^2 & q \\ 1 & v & v^2 & r \end{array} \right] &\xrightarrow{\substack{R_2 := R_2 - R_1 \\ R_3 := R_3 - R_1}} \left[\begin{array}{ccc|c} 1 & t & t^2 & p \\ 0 & u - t & u^2 - t^2 & q - p \\ 0 & v - t & v^2 - t^2 & r - p \end{array} \right] \\ &\xrightarrow{\substack{R_2 := \frac{1}{u-t} R_2 \\ R_3 := \frac{1}{v-t} R_3}} \left[\begin{array}{ccc|c} 1 & t & t^2 & p \\ 0 & 1 & u + t & \frac{q-p}{u-t} \\ 0 & 1 & v + t & \frac{r-p}{v-t} \end{array} \right] \\ &\xrightarrow{R_3 := R_3 - R_2} \left[\begin{array}{ccc|c} 1 & t & t^2 & p \\ 0 & 1 & u + t & \frac{q-p}{u-t} \\ 0 & 0 & v - u & \frac{r-p}{v-t} - \frac{q-p}{u-t} \end{array} \right] \\ &\xrightarrow{R_3 := \frac{1}{v-u} R_3} \left[\begin{array}{ccc|c} 1 & t & t^2 & p \\ 0 & 1 & u + t & \frac{q-p}{u-t} \\ 0 & 0 & 1 & * \end{array} \right] \end{aligned}$$

where I have not bothered to compute the final entry, because the question did not require us to find the solution, but merely to show that a solution exists, and this is already clear from what we have done. Note that the row operations that involved dividing by $u - t$, $v - t$ and $v - u$ are justified by the fact that t , u and v are distinct real numbers (which we were given), so that none of $u - t$, $v - t$ and $v - u$ are zero.

Let a , b and c solve the given system of equations. The equation $p = a + bt + ct^2$ can be interpreted as saying that graph of $y = a + bx + cx^2$ passes through the point (t, p) , and similarly the other two equations say that (u, q) and (v, r) lie on this same graph. (Note that the graph is a parabola if $a \neq 0$ and a straight line if $a = 0$.)

Answers to Preparatory Questions

1. $R_2 := R_2 + \frac{7}{4}R_1$

2. (i) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 4 & -1 & 8 & 3 \\ -1 & 3 & -3 & 2 \end{array} \right]$ (ii) $\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 11 & -2 & 8 & 0 \\ 1 & 1 & -1 & 0 \end{array} \right]$

3. In the new matrix, the second row will be the negative of the first row and in general information will be lost. For example, $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ would become $\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$ and all subsequent row operations would leave the matrix with 0's in the second column.

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

www.maths.usyd.edu.au/u/UG/JM/MATH1902/

and then do the Web Quiz for Week 6.