The University of Sydney

MATH1902 Linear Algebra (advanced)

Exercises for Week 7 (beginning April 20) Semester 1 2009

Preliminary Reading:

Chapter 2 of the Linear Algebra book.

Objectives:

By the end of Week 7, to achieve at least a pass level, you should be able to

7A: compute sums and products of matrices,

7B: compute with row and column vectors,

7C: reduce matrices to row echelon form and to reduced row echelon form.

To achieve higher than a pass level you should be able to

7D: use the "sigma notation" for summation of series,

7E: prove the basic laws of matrix algebra,

7F: find all matrices satisfying given conditions.

Preparatory questions. (Answers are on the next page.)

1. Given the matrices $C = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 5 \\ 6 & 1 \end{bmatrix}$, find $C + D$, $3C$, CD and DC.

2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -6 \\ 7 & 12 & 17 \end{bmatrix}$. Use elementary row operations to transform A to a **reduced** row echelon matrix.

3. Write the column vector $\begin{bmatrix} 2+s+3t \\ 1-t \\ s \\ -5 \end{bmatrix}$ as a sum of the form $\mathbf{a} + s\mathbf{b} + t\mathbf{c}$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are column vectors.

Practice questions

4. Let $A = \begin{bmatrix} -2 & 1 & 1 \\ -2 & 0 & 1 \\ -4 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 0 & 2 \\ 4 & 1 & -2 \\ 6 & 0 & -2 \end{bmatrix}$. Compute A^3, BA and AB.

Solution.

A^3 is the zero matrix, $BA = A$ and $AB = \begin{bmatrix} 16 & 1 & -8 \\ 12 & 0 & -6 \\ 32 & 2 & -16 \end{bmatrix}$.

5. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$. Suppose that $AD = DA$, and suppose also that no two of a, b, c are equal. Show that a_{12}, a_{21}, a_{13}, a_{31}, a_{23} and a_{32} are all zero.

Solution.

The $(1,2)$-entry of AD is $a_{12}b$, while the $(1,2)$-entry of DA is aa_{12}. Since $AD = DA$ it follows that $a_{12}b = aa_{12}$, and so $(a-b)a_{12} = 0$. Since $a - b \neq 0$ it follows that $a_{12} = 0$. The other conclusions follow similarly by considering the other off-diagonal elements of $AD = DA$.

6. Prove that $(A+B)C = AC + BC$ whenever A and B are $r \times n$ matrices and C is an $n \times p$ matrix.

Solution.

Let a_{ij}, b_{ij}, c_{ij} denote the (i,j)th entries of A, B, C respectively.

Consider any integers i and j with $1 \leq i \leq n$ and $1 \leq j \leq p$. The (i,j)th entries of AC and BC are

$$\sum_{k=1}^{n} a_{ik}c_{kj} \qquad \text{and} \qquad \sum_{k=1}^{n} b_{ik}c_{kj}$$

respectively. Hence the (i,j)th entry of $AB + AC$ is $\sum_{k=1}^{n} a_{ik}c_{kj} + \sum_{k=1}^{n} b_{ik}c_{kj}$. This equals

$$\sum_{k=1}^{n}(a_{ik}c_{kj} + b_{kj}c_{kj}) = \sum_{k=1}^{n}(a_{ik} + b_{kj})c_{kj}$$

which is the (i,j)th entry of $(A+B)C$. This reasoning applies for all values of i and j; so we have shown the every entry of $AB + AC$ equals the corresponding entry of $(A+B)C$. Hence $AC + BC = (A+B)C$.

7. Let $\mathbf{x}$ be a 1×3 row vector and $\mathbf{y}$ a 3×1 column vector. Given that

$$\mathbf{yx} = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & -6 \\ 6 & -3 & 9 \end{bmatrix}.$$

determine $\mathbf{xy}$.

Solution.

Suppose that $\mathbf{x} = [x_1 \quad x_2 \quad x_3]$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. The product of a 3×1 matrix by a 1×3 matrix gives a 3×3 matrix, and indeed we find that

$$\mathbf{yx} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} [x_1 \quad x_2 \quad x_3] = \begin{bmatrix} y_1x_1 & y_1x_2 & y_1x_3 \\ y_2x_1 & y_2x_2 & y_2x_3 \\ y_3x_1 & y_3x_2 & y_3x_3 \end{bmatrix}.$$

Multiplying a 1×3 matrix by a 3×1 matrix gives a 1×1 matrix—that is, just a number—and we find that

$$\mathbf{xy} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1y_1 + x_2y_2 + x_3y_3.$$

So, in fact, the number $\mathbf{xy}$ is the sum of the $(1,1)$, $(2,2)$ and $(3,3)$ entries of the matrix $\mathbf{yx}$. With $\mathbf{yx}$ as given above, we see that $y_1x_1 = 2$, $y_2x_2 = 2$ and $y_3x_3 = 9$, and therefore $\mathbf{xy} = 2 + 2 + 9 = 13$.

We should really check that it is possible to find $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{yx}$ equals the given matrix, and indeed it is readily checked that $\mathbf{y} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\mathbf{x} = [2, -1, 3]$ will do. This answer is not unique: for example, halving all the entries of $\mathbf{x}$ and doubling all the entries of $\mathbf{y}$ would give another solution.

8. Express $\mathbf{v} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$. That is, find numbers x, y and z such that $\mathbf{v} = x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3$.

Solution.

We want $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} x + y + 2z \\ x + 3y - z \\ 5x - 2y + 4z \end{bmatrix}$ and so we want to solve the following system of linear equations:

$$\begin{aligned} x + y + 2z &= 5 \\ x + 3y - z &= 3 \\ 5x - 2y + 4z &= 1. \end{aligned}$$

As usual, form the augmented matrix and apply row operations:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 1 & 3 & -1 & 3 \\ 5 & -2 & 4 & 1 \end{array}\right] \xrightarrow[R_3:=R_3-5R_1]{R_2:=R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 2 & -3 & -2 \\ 0 & -7 & -6 & -24 \end{array}\right] \xrightarrow{R_2:=(1/2)R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & -\frac{3}{2} & -1 \\ 0 & -7 & -6 & -24 \end{array}\right]$$

$$\xrightarrow{R_3:=R_3+7R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & -\frac{3}{2} & -1 \\ 0 & 0 & -\frac{33}{2} & -31 \end{array}\right] \xrightarrow{R_3:=(2/33)R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & -\frac{3}{2} & -1 \\ 0 & 0 & 1 & \frac{62}{33} \end{array}\right] \longrightarrow$$

$$\xrightarrow[R_1:=R_1-2R_3]{R_2:=R_2+(3/2)R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{41}{33} \\ 0 & 1 & 0 & \frac{20}{11} \\ 0 & 0 & 1 & \frac{62}{33} \end{array}\right] \xrightarrow[R_1:=R_1-2R_3]{R_1:=R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{19}{33} \\ 0 & 1 & 0 & \frac{20}{11} \\ 0 & 0 & 1 & \frac{62}{33} \end{array}\right]$$

Thus we conclude that $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = -\frac{19}{33}\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + \frac{20}{11}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{62}{33}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$.

Answers to Preparatory Questions

1. $C + D = \begin{bmatrix} 6 & 4 \\ 9 & 3 \end{bmatrix}$, $3C = \begin{bmatrix} 6 & -3 \\ 9 & 6 \end{bmatrix}$, $CD = \begin{bmatrix} 2 & 9 \\ 24 & 17 \end{bmatrix}$, and $DC = \begin{bmatrix} 23 & 6 \\ 15 & -4 \end{bmatrix}$.

2. The row operations are

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & -6 \\ 7 & 12 & 17 \end{bmatrix} \xrightarrow[R_3:=R_3-7R_1]{R_2:=R_2+2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & -4 \\ 0 & 5 & 10 \end{bmatrix} \xrightarrow{R_2:=\frac{1}{4}R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 5 & 10 \end{bmatrix}$$

$$\xrightarrow{R_3:=R_3-5R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 15 \end{bmatrix} \xrightarrow{R_3:=\frac{1}{15}R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2:=R_2-R_3]{R_1:=R_1+R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1:=R_1-R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.

$$\begin{bmatrix} 2+s+3t \\ 1-t \\ s \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -5 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 7.