The University of Sydney

MATH1902 Linear Algebra (advanced)

Exercises for Week 8 (beginning April 27) Semester 1 2009

Preliminary Reading:

Chapter 2 of the Linear Algebra book.

Objectives:

By the end of Week 8, to achieve at least a pass level, you should be able to

8A: find the inverse of a 2×2 matrix by direct calculation,

8B: manipulate expressions given in $\sum$-notation,

8C: use row operations to compute the inverse of a matrix.

To achieve higher than a pass level you should be able to

8D: relate elementary row operations and elementary matrices,

8E: employ elementary matrices to prove facts about matrices in general,

8F: solve simple problems involving "abstract" matrices.

Preparatory questions. (Answers are on the next page.)

1. Find the inverse of the matrix $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$.

2. Simplify the expression $\sum_{i=1}^{3} \sum_{j=1}^{2} \sum_{k=1}^{2} a_{ij}\delta_{jk}b_{ki}$, where δ_{jk} is the Kronecker delta.

3. Find the inverse of $\begin{bmatrix} 1 & -2 & -1 \\ -3 & 5 & 1 \\ 10 & -12 & 8 \end{bmatrix}$.

4. Find the inverse of $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & -3 & -7 \end{bmatrix}$.

Practice questions

5. For each of the following matrices, determine whether it is invertible, and find the inverse if there is one.

(i) $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & 2 \\ -3 & 0 & 3 \\ -2 & -1 & 0 \end{bmatrix}$

Solution.

To decide whether an $n \times n$ matrix A is invertible, form the $n \times 2n$ matrix $[A \mid I]$ and apply row operations to produce a reduced echelon matrix. If the reduced echelon matrix has the form $[I \mid B]$ then A is invertible and $B = A^{-1}$. If, on the other hand, row operations give a matrix of the form $[J \mid B]$, where J has a row of zeros, then A is not invertible.

(i)

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3:=R_3-R_1]{R_2:=R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & -7 & 1 & -2 & 0 \\ 0 & 2 & -2 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & -1 & 1 \\ 0 & 0 & -7 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[R_3:=\frac{-1}{7}R_3]{R_2:=\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{2}{7} & 0 \end{array}\right] \xrightarrow[R_2:=R_2+R_3]{R_1:=R_1-3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & \frac{1}{7} & 0 \\ 0 & 1 & 0 & -\frac{1}{7} & -\frac{3}{14} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{7} & \frac{2}{7} & 0 \end{array}\right].$$

Thus the matrix is invertible, with inverse $\begin{bmatrix} \frac{3}{7} & \frac{1}{7} & 0 \\ -\frac{1}{7} & -\frac{3}{14} & \frac{1}{2} \\ -\frac{1}{7} & \frac{2}{7} & 0 \end{bmatrix}$.

(ii)

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ -3 & 0 & 3 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} -3 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1:=-\frac{1}{3}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3:=R_3+2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & -\frac{2}{3} & 1 \end{array}\right]$$

$$\xrightarrow{R_3:=R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{2}{3} & 1 \end{array}\right].$$

Since we have produced a row of zeros in the left hand half of the matrix, we need go no farther. The reduced echelon matrix will not be of the form $[I \mid X]$, and so the given matrix was not invertible.

6. Use the previous question to solve the following system of linear equations:

$$\begin{aligned} 2x - z &= 2 \\ x + 3z &= 1 \\ x + 2y + z &= -1 \end{aligned}$$

Solution.

If A is a square matrix with an inverse then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. So by Exercise 5$(i)$ the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} & 0 \\ -\frac{1}{7} & -\frac{3}{14} & \frac{1}{2} \\ -\frac{1}{7} & \frac{2}{7} & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

That is, $x = 1$, $y = -1$, $z = 0$.

7. Use elementary row operations to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 6 & 3 & 0 \end{bmatrix},$$

and hence express A as a product of elementary matrices. (From 1996 exam.)

Solution.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 6 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 := R_3 - 6R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -12 & -6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\substack{R_1 := R_1 - 2R_2 \\ R_3 := R_3 + 12R_2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & -6 & 12 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 3 & 0 & -6 & 12 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 := \frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 & 4 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

This shows that $A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & \frac{1}{3} \\ 0 & 1 & 0 \end{bmatrix}$.

The elementary matrices corresponding to the row operations that were used, in the order in which they were used, are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 12 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If we call these E_1, E_2, E_3, E_4 and E_5 (respectively) then we have

$$E_5 E_4 E_3 E_2 E_1 [A \mid I] = [I \mid A^{-1}]$$

which shows that $E_5 E_4 E_3 E_2 E_1 = A^{-1}$. Hence $A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1}$. That is,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -12 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This answer is not unique because there is more than one way to carry out the row operations.

8. Let X be a square matrix and suppose that for some real numbers $t_0, t_1, \ldots, t_n$,

$$t_n X^n + t_{n-1} X^{n-1} + \cdots + t_1 X + t_0 I = \mathbf{0}.$$

Show that if $t_0 \neq 0$ then X is invertible. (Hint: move the term $t_0 I$ to the right hand side and factorise the left hand side.)

Solution.

Rearranging as suggested in the hint we find that

$$(-\tfrac{t_n}{t_0}X^{n-1}-\tfrac{t_{n-1}}{t_0}X^{n-2}-\cdots-\tfrac{t_1}{t_0}I)X = I = X(-\tfrac{t_n}{t_0}X^{n-1}-\tfrac{t_{n-1}}{t_0}X^{n-2}-\cdots-\tfrac{t_1}{t_0}I),$$

and so $(-\frac{t_n}{t_0}X^{n-1}-\frac{t_{n-1}}{t_0}X^{n-2}-\cdots-\frac{t_1}{t_0}I)$ is an inverse, and hence *the* inverse, of X.

9. Let J be the $n \times n$ matrix each of whose entries is 1.

(*i*) Show that $J^2 = nJ$.

(*ii*) Show that if $n > 1$ then $(I - J)^{-1} = I - \dfrac{1}{n-1}J$.

Solution.

(*i*) Each entry of J^2 is the product of the n-component row of 1's by the n-component column of 1's, which gives the sum of n 1's, which is n. So each entry of J^2 is n times the corresponding entry of J; that is, $J^2 = nJ$.

(*ii*)

$$\begin{aligned}(I-J)(I-\frac{1}{n-1}J) &= (I-\frac{1}{n-1}J)(I-J)\\ &= I-\frac{1}{n-1}J-J+\frac{1}{n-1}J^2\\ &= I-\frac{n}{n-1}J+\frac{1}{n-1}(nJ) = I.\end{aligned}$$

By the definition, this shows that $I-\frac{1}{n-1}J$ is an inverse of $I-J$, and since a matrix can have at most one inverse, we have $(I-J)^{-1} = I-\frac{1}{n-1}J$.

10. Let A be an $n \times n$ matrix whose (i,j)-th entry is a_{ij}. We say that A is *upper triangular* if $a_{ij} = 0$ whenever $i > j$. Show that the product of two $n \times n$ upper triangular matrices is upper triangular. (Do the case $n = 3$ before attempting the general case.)

Solution.

Let a_{ij} be the (i,j)th entry of A and b_{ij} be the (i,j)th entry of B. Then the (i,j)th entry of AB is $\sum_{k=1}^{n} a_{ik}b_{kj}$. Suppose now that A and B are upper triangular, so that $a_{ik} = 0$ whenever $i > k$ and $b_{kj} = 0$ whenever $k > j$. Then if $i > j$ we have

$$\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{j} a_{ik}b_{kj} + \sum_{k=j+1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{j} 0b_{kj} + \sum_{k=j+1}^{n} a_{ik}0 = 0+0 = 0,$$

since when $1 \leq k \leq j$ we have $i > k$ (since $i > j$) and hence $a_{ik} = 0$, and when $j+1 \leq k \leq n$ we have $k > j$ and hence $b_{kj} = 0$. So the (i,j)th entry of AB is zero when $i > j$, as required.

It is easier to see this in an example. The (2,1)-entry of

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \begin{pmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & l \end{pmatrix}$$

is $[0 \quad d \quad e] \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix} = 0g + d0 + e0 = 0$, and the other below-diagonal entries similarly can be seen to be zero.

Answers to Preparatory Questions

1. $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$.

2. $a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22} + a_{31}b_{13} + a_{32}b_{23}$.

3. The inverse is $\begin{bmatrix} -26 & -14 & -\frac{3}{2} \\ -17 & -9 & -1 \\ 7 & 4 & \frac{1}{2} \end{bmatrix}$, as shown by the following sequence of elementary row operations:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 & 1 & 0 \\ 10 & -12 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3:=R_3-10R_1]{R_2:=R_2+3R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 3 & 1 & 0 \\ 0 & 8 & 18 & -10 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2:=(-1)R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & -1 & 0 \\ 0 & 8 & 18 & -10 & 0 & 1 \end{array}\right] \xrightarrow{R_3:=R_3-8R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & -1 & 0 \\ 0 & 0 & 2 & 14 & 8 & 1 \end{array}\right]$$

$$\xrightarrow{R_3:=(1/2)R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & -1 & 0 \\ 0 & 0 & 1 & 7 & 4 & \frac{1}{2} \end{array}\right] \xrightarrow[R_1:=R_1+R_3]{R_2:=R_2-2R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 8 & 4 & \frac{1}{2} \\ 0 & 1 & 0 & -17 & -9 & -1 \\ 0 & 0 & 1 & 7 & 4 & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_1:=R_1+2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -26 & -14 & -\frac{3}{2} \\ 0 & 1 & 0 & -17 & -9 & -1 \\ 0 & 0 & 1 & 7 & 4 & \frac{1}{2} \end{array}\right]$$

4. Note that the 2×2 blocks in the inverse are the inverses of the 2×2 blocks in the given matrix and so the inverse is $\begin{bmatrix} 3 & -2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 3 & 2 \end{bmatrix}$.

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 8.