THE UNIVERSITY OF SYDNEY
MATH1902 Linear Algebra (advanced)

Exercises for Week 9 (beginning May 4) Semester 1 2009

Preliminary Reading:

Chapter 2 of the Linear Algebra book.

Objectives:

By the end of Week 9, to achieve at least a pass level, you should be able to

9A: use row operations to compute the inverse of a matrix,
9B: find the parity of a permutation,
9C: write a permutation in cycle form.

To achieve higher than a pass level you should be able to

9D: relate elementary row operations and elementary matrices,
9E: understand the connection between finding the inverse of a matrix and solving systems of simultaneous linear equations,
9F: compose permutations given either in two-line form or in cycle form.

Preparatory questions. (Answers are on the next page.)

1. Find the inverse of the matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

2. Find the inverse of $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 4 & 2 \end{bmatrix}$, avoiding fractions in your calculations as long as possible.

3. Given the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$,

(i) Determine the parity of σ.

(ii) Write σ as a product of disjoint cycles.

Practice questions

4. Let $A = \begin{bmatrix} 1 & -2 & -1 \\ -3 & 5 & 1 \\ 10 & -12 & 8 \end{bmatrix}$.

(i) Use elementary row operations to transform $[\,A \mid I\,]$ to a reduced echelon matrix $[\,I \mid B\,]$, and hence show that A is invertible and find its inverse. Record the row operations you use, and make use of this information to express the matrix A as a product of elementary matrices.

(ii) Use the inverse of A (calculated in (i)) to solve the following matrix equations for X, Y and Z:

$$AX = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad AY = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 4 & 3 \end{bmatrix}, \qquad ZA = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}.$$

Solution.

(*i*)
$$\left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 & 1 & 0 \\ 10 & -12 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3:=R_3-10R_1]{R_2:=R_2+3R_1} \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 3 & 1 & 0 \\ 0 & 8 & 18 & -10 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2:=(-1)R_2} \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & -1 & 0 \\ 0 & 8 & 18 & -10 & 0 & 1 \end{array}\right] \xrightarrow{R_3:=R_3-8R_2} \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & -1 & 0 \\ 0 & 0 & 2 & 14 & 8 & 1 \end{array}\right]$$

$$\xrightarrow{R_3:=(1/2)R_3} \left[\begin{array}{rrr|rrr} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3 & -1 & 0 \\ 0 & 0 & 1 & 7 & 4 & \frac{1}{2} \end{array}\right] \xrightarrow[R_1:=R_1+R_3]{R_2:=R_2-2R_3} \left[\begin{array}{rrr|rrr} 1 & -2 & 0 & 8 & 4 & \frac{1}{2} \\ 0 & 1 & 0 & -17 & -9 & -1 \\ 0 & 0 & 1 & 7 & 4 & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_1:=R_1+2R_2} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -26 & -14 & -\frac{3}{2} \\ 0 & 1 & 0 & -17 & -9 & -1 \\ 0 & 0 & 1 & 7 & 4 & \frac{1}{2} \end{array}\right].$$

So $A^{-1} = \begin{bmatrix} -26 & -14 & -\frac{3}{2} \\ -17 & -9 & -1 \\ 7 & 4 & \frac{1}{2} \end{bmatrix}$.

Let E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8 be the elementary matrices corresponding to the row operations used, in the order in which they were used. Then one way of expressing A as a product of elementary matrices is

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}E_8^{-1}.$$

Note that the inverse of the elementary matrix which corresponds to a given row operation is the elementary matrix corresponding to the reverse row operation. Using this we find that the E_i^{-1}'s are as follows:

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 10 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 8 & 1 \end{bmatrix}, \quad E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad E_6^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_7^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_8^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(*ii*) The solutions are:

$$X = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -26 & -14 & -\frac{3}{2} \\ -17 & -9 & -1 \\ 7 & 4 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{83}{2} \\ -27 \\ \frac{23}{2} \end{bmatrix},$$

$$Y = A^{-1}\begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -26 & -14 & -\frac{3}{2} \\ -17 & -9 & -1 \\ 7 & 4 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -18 & \frac{57}{2} \\ -11 & -19 \\ 2 & \frac{15}{2} \end{bmatrix},$$

$$Z = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} A^{-1} = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -26 & -14 & -\frac{3}{2} \\ -17 & -9 & -1 \\ 7 & 4 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 40 & 22 & \frac{5}{2} \end{bmatrix}.$$

5. After applying the following sequence of elementary row operations: $R_1 \leftrightarrow R_3$, $R_2 := R_2 - 2R_1$, $R_3 := R_3 - R_2$, $R_3 := -R_3$, $R_2 := R_2 - 2R_3$, $R_1 := R_1 + 7R_3$, $R_1 := R_1 - 4R_2$ to a certain 3×3 matrix A, the resulting matrix was the identity.

(i) Write down the elementary matrices corresponding to the given row operations.

(ii) Write down the inverses of the elementary matrices in (i).

(iii) Write down expressions for A and A^{-1} as products of the elementary matrices from (i) or (ii) and hence calculate A and A^{-1}.

Solution.

(i) Let E_i be the elementary matrix corresponding to the i-th row operation that was applied. Listed in order, the E_i are as follows:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(ii) Their inverses are (respectively)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(iii) To obtain the inverse one applies the same row operations in the same order to the identity matrix. Note that the first row operation transforms I to $E_1 I$, the next transforms $E_1 I$ to $E_2 E_1 I$, and so on, so that the final answer is $A^{-1} = E_7 E_6 E_5 E_4 E_3 E_2 E_1$. But the easiest way to calculate it is by doing the row operations.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 := R_2 - 2R_1} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 := R_3 - R_2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 2 \end{bmatrix}$$
$$\xrightarrow{R_3 := -R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix} \xrightarrow[R_1 := R_1 + 7R_3]{R_2 := R_2 - 2R_3} \begin{bmatrix} -7 & 7 & -13 \\ 2 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix} \xrightarrow{R_1 := R_1 - 4R_2} \begin{bmatrix} -15 & 11 & -21 \\ 2 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

Similarly, the matrix A is $E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1}$, calculated by applying the reverse row operations to the identity matrix, in the reverse order.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 := R_1 + 4R_2} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 := R_1 - 7R_3]{R_2 := R_2 + 2R_3} \begin{bmatrix} 1 & 4 & -7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 := -R_3} \begin{bmatrix} 1 & 4 & -7 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$
$$\xrightarrow{R_3 := R_3 + R_2} \begin{bmatrix} 1 & 4 & -7 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 := R_2 + 2R_1} \begin{bmatrix} 1 & 4 & -7 \\ 2 & 9 & -12 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 9 & -12 \\ 1 & 4 & -7 \end{bmatrix}$$

6. Consider the two systems of linear equations:

$$\begin{aligned} x + 2y + 2z &= 3 \\ y + 3z &= -1 \\ z &= 4 \end{aligned} \qquad \begin{aligned} x + 2y + 2z &= 2 \\ y + 3z &= 2 \\ z &= 2 \end{aligned}$$

Observe that they have the same coefficient matrix. Solve them both at once by applying row operations to the augmented matrix

$$\left[\begin{array}{ccc|cc} 1 & 2 & 2 & 3 & 2 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right].$$

Solution.

$$\left[\begin{array}{ccc|cc} 1 & 2 & 2 & 3 & 2 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right] \xrightarrow[R_2:=R_2-3R_3]{R_1:=R_1-2R_3} \left[\begin{array}{ccc|cc} 1 & 2 & 0 & -5 & -2 \\ 0 & 1 & 0 & -13 & -4 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right]$$

$$\xrightarrow{R_1:=R_1-2R_2} \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 21 & 6 \\ 0 & 1 & 0 & -13 & -4 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right]$$

Thus the first system has unique solution $x = 21$, $y = -13$, $z = 4$, and the second system has unique solution $x = 6$, $y = -4$, $z = 2$.

7. Given a certain 3×3 matrix A and the three systems of equations

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Suppose that the first system has solution $x = 1$, $y = 1$, $z = 0$, the second has solution $x = 0$, $y = 1$, $z = 3$, and the third has solution $x = y = z = 1$. What is the inverse of the matrix A?

Solution.

We are told that

$$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Combining these into a single matrix equation gives

$$A \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$.

8. (*i*) Let A be an 3×3 matrix, and suppose that $[1 \;\; 1 \;\; 1]\,A = [0 \;\; 0 \;\; 0]$. Show that A is not invertible. (Hint: suppose that A^{-1} exists, and obtain a contradiction by considering $[1 \;\; 1 \;\; 1]\,AA^{-1}$.)

(*ii*) Let A be an $n \times n$ matrix, and suppose that there exists a nonzero $1 \times n$ matrix (row vector) B such that $BA = \mathbf{0}$ (where $\mathbf{0}$ is the $1 \times n$ zero matrix). Show that A is not invertible. Similarly, show that if there is a nonzero $n \times 1$ matrix (column vector) C such that AC is zero then A is not invertible.

Solution.

(*i*) If A^{-1} exists then

$$\begin{aligned}[1 \;\; 1 \;\; 1] &= [1 \;\; 1 \;\; 1]\,(AA^{-1}) = ([1 \;\; 1 \;\; 1]\,A)A^{-1} \\ &= [0 \;\; 0 \;\; 0]\,A^{-1} = [0 \;\; 0 \;\; 0]\,,\end{aligned}$$

a contradiction.

(*ii*) Suppose that A^{-1} exists. Then $B = BI = B(AA^{-1}) = (BA)A^{-1} = \mathbf{0}A^{-1} = \mathbf{0}$, contradiction. Similarly if C is a nonzero column matrix such that $AC = \mathbf{0}$ then $C = A^{-1}AC = A^{-1}\mathbf{0} = \mathbf{0}$, contradiction. So A cannot be invertible if there is such a B or such a C.

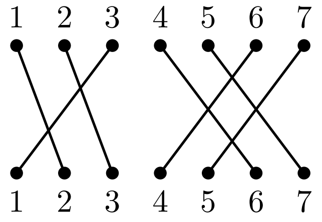
9. Draw diagrams for the following permutations of 1, 2, 3, 4, 5, 6, 7, and hence determine whether they are odd or even.

(*i*) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 6 & 7 & 4 & 5 \end{pmatrix}$ (*ii*) $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 7 & 4 & 5 & 6 & 2 \end{pmatrix}$

(*iii*) $\sigma\tau$ (*iv*) $\tau\sigma$

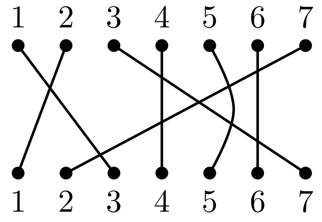
Solution.

(*i*)



$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 2\,3\,1\,6\,7\,4\,5 \end{pmatrix}$

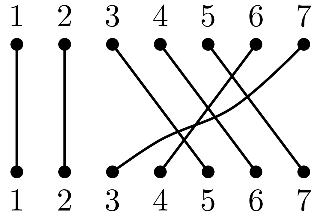
(*ii*)



$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 3\,1\,7\,4\,5\,6\,2 \end{pmatrix}$

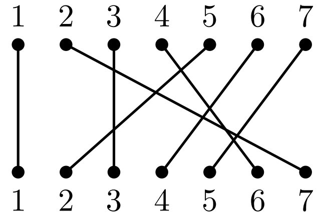
In (*i*) there are 6 crossings in the diagram, so that the permutation is even. In (*ii*) there are 9 crossings, and so the permutation is odd.

(*i*)



$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,2\,5\,6\,7\,4\,3 \end{pmatrix}$

(*ii*)



$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 1\,7\,3\,6\,2\,4\,5 \end{pmatrix}$

In (*iii*) there are 7 crossings in the diagram, so that the permutation is odd. In (*iv*) there are 9 crossings, and so the permutation is odd.

10. (*i*) Calculate the product of the following two matrices:

$$A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & p & q \\ 0 & 1 & r & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

What is the inverse of A?

(*ii*) Let A, B, C, D, E, F, G and H be 2×2 matrices, and use them to construct two 4×4 "block matrices" P and Q as follows:

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad Q = \begin{bmatrix} E & F \\ G & H \end{bmatrix}.$$

Show that $PQ = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$.

Solution.

(*i*) We find that AB and BA are both equal to $\begin{bmatrix} 1 & 0 & a+p & b+q \\ 0 & 1 & c+r & d+s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Thus $AB = BA = I$ if $p = -a$, $q = -b$, $r = -c$ and $s = -d$, and therefore the inverse of A is

$$A^{-1} = \begin{bmatrix} 1 & 0 & -a & -b \\ 0 & 1 & -c & -d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(*ii*) Let a_{ij} be the (i,j)-entry of A, let b_{ij} be the (i,j)-entry of B, and so on. Thus

$$PQ = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & f_{11} & f_{12} \\ e_{21} & e_{22} & f_{21} & f_{22} \\ g_{11} & g_{12} & h_{11} & h_{12} \\ g_{21} & g_{22} & h_{21} & h_{22} \end{bmatrix}.$$

The $(1,1)$-entry of the product is obtained by multiplying the first row of P by the first column of Q. It is thus $a_{11}e_{11} + a_{12}e_{21} + b_{11}g_{11} + b_{12}g_{21}$, and the other entries can be found similarly. Partitioning PQ into 2×2 blocks, we can write $PQ = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$, where X, Y, Z and W are as follows:

$$\begin{aligned} X &= \begin{bmatrix} a_{11}e_{11} + a_{12}e_{21} + b_{11}g_{11} + b_{12}g_{21} & a_{11}e_{12} + a_{12}e_{22} + b_{11}g_{12} + b_{12}g_{22} \\ a_{21}e_{11} + a_{22}e_{21} + b_{21}g_{11} + b_{22}g_{21} & a_{21}e_{12} + a_{22}e_{22} + b_{21}g_{12} + b_{22}g_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}e_{11} + a_{12}e_{21} & a_{11}e_{12} + a_{12}e_{22} \\ a_{21}e_{11} + a_{22}e_{21} & a_{21}e_{12} + a_{22}e_{22} \end{bmatrix} + \begin{bmatrix} b_{11}g_{11} + b_{12}g_{21} & b_{11}g_{12} + b_{12}g_{22} \\ b_{21}g_{11} + b_{22}g_{21} & b_{21}g_{12} + b_{22}g_{22} \end{bmatrix} \\ &= AE + BG, \end{aligned}$$

$$\begin{aligned}
Y &= \begin{bmatrix} a_{11}f_{11}+a_{12}f_{21}+b_{11}h_{11}+b_{12}h_{21} & a_{11}f_{12}+a_{12}f_{22}+b_{11}h_{12}+b_{12}h_{22} \\ a_{21}f_{11}+a_{22}f_{21}+b_{21}h_{11}+b_{22}h_{21} & a_{21}f_{12}+a_{22}f_{22}+b_{21}h_{12}+b_{22}h_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}f_{11}+a_{12}f_{21} & a_{11}f_{12}+a_{12}f_{22} \\ a_{21}f_{11}+a_{22}f_{21} & a_{21}f_{12}+a_{22}f_{22} \end{bmatrix} + \begin{bmatrix} b_{11}h_{11}+b_{12}h_{21} & b_{11}h_{12}+b_{12}h_{22} \\ b_{21}h_{11}+b_{22}h_{21} & b_{21}h_{12}+b_{22}h_{22} \end{bmatrix} \\
&= AF+BH, \\
Z &= \begin{bmatrix} c_{11}e_{11}+c_{12}e_{21}+d_{11}g_{11}+d_{12}g_{21} & c_{11}e_{12}+c_{12}e_{22}+d_{11}g_{12}+d_{12}g_{22} \\ c_{21}e_{11}+c_{22}e_{21}+d_{21}g_{11}+d_{22}g_{21} & c_{21}e_{12}+c_{22}e_{22}+d_{21}g_{12}+d_{22}g_{22} \end{bmatrix} \\
&= \begin{bmatrix} c_{11}e_{11}+c_{12}e_{21} & c_{11}e_{12}+c_{12}e_{22} \\ c_{21}e_{11}+c_{22}e_{21} & c_{21}e_{12}+c_{22}e_{22} \end{bmatrix} + \begin{bmatrix} d_{11}g_{11}+d_{12}g_{21} & d_{11}g_{12}+d_{12}g_{22} \\ d_{21}g_{11}+d_{22}g_{21} & d_{21}g_{12}+d_{22}g_{22} \end{bmatrix} \\
&= CE+DG, \\
W &= \begin{bmatrix} c_{11}f_{11}+c_{12}f_{21}+d_{11}h_{11}+d_{12}h_{21} & c_{11}f_{12}+c_{12}f_{22}+d_{11}h_{12}+d_{12}h_{22} \\ c_{21}f_{11}+c_{22}f_{21}+d_{21}h_{11}+d_{22}h_{21} & c_{21}f_{12}+c_{22}f_{22}+d_{21}h_{12}+d_{22}h_{22} \end{bmatrix} \\
&= \begin{bmatrix} c_{11}f_{11}+c_{12}f_{21} & c_{11}f_{12}+c_{12}f_{22} \\ c_{21}f_{11}+c_{22}f_{21} & c_{21}f_{12}+c_{22}f_{22} \end{bmatrix} + \begin{bmatrix} d_{11}h_{11}+d_{12}h_{21} & d_{11}h_{12}+d_{12}h_{22} \\ d_{21}h_{11}+d_{22}h_{21} & d_{21}h_{12}+d_{22}h_{22} \end{bmatrix} \\
&= CF+DH.
\end{aligned}$$

Answers to Preparatory Questions

1. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

2. The inverse is $\begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} \\ -\frac{5}{3} & -1 & \frac{7}{3} \\ \frac{7}{3} & 2 & -\frac{11}{3} \end{bmatrix}$, as shown by the following row operations

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1:=R_1-R_2} \left[\begin{array}{ccc|ccc} 1 & -3 & -2 & 1 & -1 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3:=R_3-3R_1]{R_2:=R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & -3 & -2 & 1 & -1 & 0 \\ 0 & 11 & 7 & -2 & 3 & 0 \\ 0 & 13 & 8 & -3 & 3 & 1 \end{array}\right] \xrightarrow{R_3:=R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & -3 & -2 & 1 & -1 & 0 \\ 0 & 11 & 7 & -2 & 3 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2:=R_2-5R_3} \left[\begin{array}{ccc|ccc} 1 & -3 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & 3 & 3 & -5 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_3:=R_3-2R_2]{R_1:=R_1+3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 10 & 8 & -15 \\ 0 & 1 & 2 & 3 & 3 & -5 \\ 0 & 0 & -3 & -7 & -6 & 11 \end{array}\right]$$

$$\xrightarrow[\substack{R_2:=R_2+\frac{2}{3}R_3 \\ R_3:=(-\frac{1}{3})R_3}]{R_1:=R_1+\frac{4}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{5}{3} & -1 & \frac{7}{3} \\ 0 & 0 & 1 & \frac{7}{3} & 2 & \frac{11}{3} \end{array}\right]$$

3. (*i*) odd (*ii*) $(1,5,3)(2,4)$

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 9.