THE UNIVERSITY OF SYDNEY

MATH1902 Linear Algebra (advanced)

Exercises for Week 10 (beginning May 11) Semester 1 2009

Preliminary Reading:

Chapter 3 of the Linear Algebra book.

Objectives:

By the end of Week 10, to achieve at least a pass level, you should be able to

10A: use row operations to solve homogeneous equations (revision),

10B: find the rank of a matrix,

10C: calculate the determinant of a matrix.

To achieve higher than a pass level you should be able to

10D: compose permutations given either in two-line form or in cycle form,

10E: calculate determinants using row expansions, elementary row operations and the formula involving parity of permutations.

Preparatory questions. (Answers are on the next page.)

1. Solve the following system of homogeneous linear equations:

$$\begin{aligned} 5x + 9y + 3z &= 0 \\ -3x + 5y + 6z &= 0 \\ -x - 5y - 3z &= 0 \end{aligned}$$

2. What is the rank of the coefficient matrix of question 1?

3. What is the determinant of the coefficient matrix of question 1?

4. Compute the determinant of $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Practice questions

5. Let $f(x_1, x_2, x_3, x_4)$ be the expression which is the product of all possible factors $x_r - x_s$ with $1 \le r < s \le 4$.

(i) Write $f(x_1, x_2, x_3, x_4)$ as a product of linear factors *without* expanding it.

(ii) Express $f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$ in terms of $f(x_1, x_2, x_3, x_4)$ when

(a) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$, (b) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$.

Solution.

(i) $f(x_1, x_2, x_3, x_4) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$

(ii) The first permutation leaves $f(x_1, x_2, x_3, x_4)$ unchanged and the second permutation changes $f(x_1, x_2, x_3, x_4)$ to its negative.

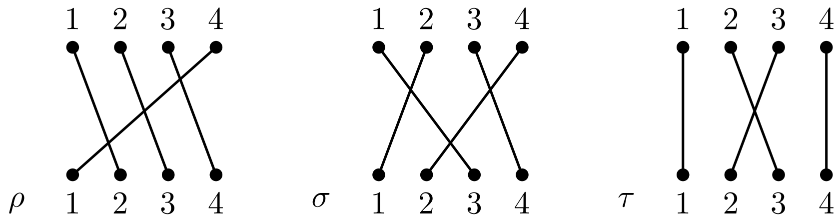
6. For the permutations

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

of $\{1, 2, 3, 4\}$, calculate $\rho \circ \sigma$, $\sigma \circ \rho$ and $\tau \circ (\sigma \circ \rho)$. For each of these permutations, determine whether it is odd or even.

Solution.

The following are diagrams for ρ, σ and τ:



Counting the number of crossings, we see that they are all odd. We have

$$(\rho \circ \sigma)(1) = \rho(\sigma(1)) = \rho(3) = 4 \qquad (\rho \circ \sigma)(2) = \rho(\sigma(2)) = \rho(1) = 2$$
$$(\rho \circ \sigma)(3) = \rho(\sigma(3)) = \rho(4) = 1 \qquad (\rho \circ \sigma)(4) = \rho(\sigma(4)) = \rho(2) = 3$$

So $\rho \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$. This is even since both ρ and σ are odd. Similarly we find that $\sigma \circ \rho = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$ (even), and $\tau \circ (\sigma \circ \rho) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ (odd).

7. Let A be an $n \times n$ matrix. Show that if the determinant of A is nonzero then $\mathbf{x} = \mathbf{0}$ is the only solution of the system of equations $A\mathbf{x} = \mathbf{0}$. By considering what happens when row operations are applied to obtain an equivalent system whose coefficient matrix is in echelon form, show that if the determinant of A is zero then $A\mathbf{x} = \mathbf{0}$ has nonzero solutions.

Solution.

Suppose that $\det A \neq 0$. This implies that A is invertible, and so if $A\mathbf{x} = \mathbf{0}$ it follows that $A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{0}$, and

$$\mathbf{x} = I\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{0} = \mathbf{0}.$$

So $\mathbf{x} = \mathbf{0}$ is the only solution of $A\mathbf{x} = \mathbf{0}$.

On the other hand, if the determinant of A is zero then the corresponding echelon matrix has a row of zeros, leaving fewer nonzero equations than there are variables in the final system of equations. There is one leading variable for each nonzero row in the echelon matrix, and since this number is less than the total number of variables it follows that some of the variables are free. Since the free variables can be given any values, there are solutions other than $\mathbf{x} = \mathbf{0}$ in this case.

8. (*i*) Show that if A is an $n \times n$ matrix and t a scalar then $\det(tA) = t^n \det A$. Deduce that $\det(-A) = -\det A$ if n is odd, while $\det(-A) = \det A$ if n is even.

(*ii*) Show that $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \det \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$. (Write out all 6 terms.)

(*iii*) Apply the result of Part (ii) to the matrix $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$, and then, using Part (i), deduce that $\det A = 0$.

Solution.

(*i*) Multiplying a row of A by t multiplies the determinant of A by t. To obtain tA from A all n rows of A must be multiplied by t. So the determinant is multiplied by t n times; that is, $\det(tA) = t^n \det A$. In particular, taking $t = -1$ gives $\det(-A) = (-1)^n \det A$, which is $+\det A$ if n is even, $-\det A$ if n is odd.

(*ii*) Using the 1st row expansion,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg) = aei - afh - bdi + bfg + cdh - ceg,$$

and similarly

$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - d\begin{vmatrix} b & h \\ c & i \end{vmatrix} + g\begin{vmatrix} b & e \\ c & f \end{vmatrix}$$

$$= a(ei - hf) - d(bi - hc) + g(bf - ec) = aei - ahf - dbi + dhc + gbf - gec,$$

which is the same.

(*iii*) Part (*ii*) above says that the transpose of a 3×3 matrix has the same determinant as the matrix itself. (The same holds for $n \times n$ matrices, for all n.) The matrix A in this part is skew-symmetric: the transpose of A equals $-A$. So $\det(-A) = \det A$ for this A. But by Part (i), $\det(-A) = (-1)^3 \det A = -\det A$, and so we conclude that $\det A = -\det A$. So $\det A = 0$. (It is also easy to write out all the terms and observe that everything cancels out.)

9. Let A and B be the following 3×3 matrices: $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$.

Calculate the determinants of both A and B

(*i*) by using the inductive definition of determinants (the "first row expansion" method), and then

(*ii*) by using the non-inductive formula for determinants, and then

(*iii*) by using the row operations method.

Solution.

(*i*) $\det A = \begin{vmatrix} 3 & 4 \\ 3 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = (6 - 12) - (2 - 4) + 2(3 - 3) = -4.$

$$\det B = \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = (1-0) - (1-0) + (4-3) = 1.$$

(ii) Writing a_{ij} for the (i,j)-entry of A, the non-inductive formula for the determinant of a 3×3 matrix gives

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

Substituting in the values we get $6 - 12 - 2 + 4 + 6 - 6 = -4$. Similarly, for B the 6 terms are $1 - 0 - 1 + 0 + 4 - 3 = 1$.

(iii) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[R_3:=R_3-R_1]{R_2:=R_2-R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_3:=R_3-R_2} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$. Since row operations of the form $R_i := R_i + \lambda R_j$ do not change determinants, and since the determinant of an upper triangular matrix is the product of the diagonal entries, we have $\det A = -4$.

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \xrightarrow[R_3:=R_3-3R_1]{R_2:=R_2-R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$ The final matrix has determinant -1, but the row swapping operation multiplies determinants by -1. So the original matrix had determinant 1.

Answers to Preparatory Questions

1. We carry out row operations on the coefficient matrix

$$\left[\begin{array}{ccc|c} 5 & 9 & 3 & 0 \\ -3 & 5 & 6 & 0 \\ -1 & -5 & -3 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & -5 & -3 & 0 \\ -3 & 5 & 6 & 0 \\ 5 & 9 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[R_3:=R_3+5R_1]{R_2:=R_2-3R_1} \left[\begin{array}{ccc|c} -1 & -5 & -3 & 0 \\ 0 & 20 & 15 & 0 \\ 0 & -16 & -12 & 0 \end{array}\right] \xrightarrow[R_3:=-\frac{1}{16}R_3]{R_2:=\frac{1}{20}R_2} \left[\begin{array}{ccc|c} -1 & -5 & -3 & 0 \\ 0 & 1 & \frac{3}{4} & 0 \\ 0 & 1 & \frac{3}{4} & 0 \end{array}\right]$$

$$\xrightarrow[R_3:=R_3+5R_1]{R_2:=R_2-3R_1} \left[\begin{array}{ccc|c} -1 & -5 & -3 & 0 \\ 0 & 1 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that the solution has one parameter, say t. To avoid fractions, set $z = 4t$ so that $y = -3t$ and $x = 3t$.

2. An echelon form of the matrix has two non-zero rows; thus the rank is 2.

3. The coefficient matrix is equivalent to one with a row of zeros; thus the determinant is 0.

4. On expanding along the first row we find that the determinant is

$$0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 2.$$

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 10.