THE UNIVERSITY OF SYDNEY

MATH1902 Linear Algebra (advanced)

Exercises for Week 11 (beginning May 18) — Semester 1 2009

Preliminary Reading:

Chapter 3 of the Linear Algebra book.

Objectives:

By the end of Week 11, to achieve at least a pass level, you should be able to

11A: write an invertible matrix as a product of elementary matrices,

11B: calculate the determinant of a matrix,

11C: recognise an eigenvector and find the associated eigenvalue of a matrix.

To achieve higher than a pass level you should be able to

11D: carry out calculations with general $m \times n$ matrices,

11E: use the sigma notation and the Kronecker delta.

Preparatory questions. (Answers are on the next page.)

1. Show that $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\begin{bmatrix} 2 & 1 & 2 \\ 3 & -4 & 4 \\ 1 & 3 & 5 \end{bmatrix}$, and determine the corresponding eigenvalue.

2. Write the matrix $A = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 2 & -3 \\ 2 & 4 & 10 \end{bmatrix}$ as a product of elementary matrices. Write down the determinant of each of the elementary matrices in this product, and hence write down the determinant of A.

Practice questions

3. Show that $\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$ is an eigenvector for $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & 5 \\ 6 & 10 & 8 \end{bmatrix}$, and determine the corresponding eigenvalue.

Solution.

$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & 5 \\ 6 & 10 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$. So the given vector is an eigenvector with eigenvalue 0.

4. Suppose that $\mathbf{v}$ is an eigenvector for the $n \times n$ matrix A with eigenvalue λ.

(i) Show that $\mathbf{v}$ is also an eigenvector for A^2 and determine the corresponding eigenvalue.

(ii) Assuming that A is invertible, show that $\mathbf{v}$ is also an eigenvector for A^{-1} and determine the corresponding eigenvalue.

Solution.

(*i*) We are given that $\mathbf{v}$ is a nonzero n-component column vector and λ a scalar satisfying $A\mathbf{v} = \lambda\mathbf{v}$. Using the basic properties of matrix algebra (see pp 48 & 50 of the Notes) we deduce that

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v},$$

which shows that $\mathbf{v}$ is an eigenvector for A with eigenvalue λ^2.

(*ii*) Multiplying both sides of the equation $A\mathbf{v} = \lambda\mathbf{v}$ on the left by A^{-1} gives $A^{-1}(A\mathbf{v}) = A^{-1}(\lambda\mathbf{v})$, and so

$$\mathbf{v} = I\mathbf{v} = (A^{-1}A)\mathbf{v} = A^{-1}(A\mathbf{v}) = A^{-1}(\lambda\mathbf{v}) = \lambda(A^{-1}\mathbf{v}). \qquad (*)$$

If λ were zero then $\lambda(A^{-1}\mathbf{v})$ would be zero, and so $\mathbf{v}$ would have to be zero. But by definition eigenvectors are nonzero; so $\mathbf{v} \neq \mathbf{0}$, and so $\lambda \neq 0$. Multiplying both sides of equation $(*)$ by $1/\lambda$ we deduce that $A^{-1}\mathbf{v} = (1/\lambda)\mathbf{v}$. Thus $\mathbf{v}$ is an eigenvector for A^{-1} with eigenvalue $1/\lambda$. (Note that in the course of this we have shown that all eigenvalues of an invertible matrix are nonzero.)

5. Let A be an $m \times n$ matrix, where $m > n$, and let D be a row echelon matrix obtained from A by applying elementary row operations. Show that D has a row of zeros, and note as a consequence that there is an invertible $m \times m$ matrix B such that BA has a row of zeros. Deduce that it is impossible to find an $n \times m$ matrix C such that $AC = I_m$.

Solution.

We shall show that the $(n+1)$-st row of D is zero. For each i from 1 to m, let k_i be the number of consecutive zeros at the start of the i-th row of D. Observe that $k_i = n$ if and only if the i-th row of D is zero (since n is the number of entries in each row of D). Now D is a row echelon matrix and so the leading nonzero entries in successive nonzero rows move further to the right as we move down the matrix. (Alternatively, the leading nonzero entries move to the left if we move up the matrix.) In other words, if $k_{i+1} < n$ (so that the the $(i+1)$-st row of D is nonzero) then $k_i \le k_{i+1} - 1$. Applying this repeatedly we see that if $k_{n+1} < n$ then

$$\begin{aligned} k_n &\le k_{n+1} - 1 < n - 1, \\ k_{n-1} &\le k_n - 1 < n - 2, \\ k_{n-2} &\le k_{n-1} < n - 3, \end{aligned}$$

and so on. To be precise, for each j we have that $k_{n-j} < n - j - 1$. But when $j = n - 1$ this gives $k_1 < 0$, which is absurd since k_1 is the number of zeros starting the first row, and thus must be at least 0. So D has a row of zeros.

Since D is obtained from A by applying row operations, $D = (E_r E_{r-1} \cdots E_1)A$ for some elementary matrices E_j. But elementary matrices are invertible and products of invertible matrices are invertible, hence $B = E_r E_{r-1} \cdots E_2 E_1$ is invertible. Suppose now that $AC = I$ for some $n \times m$ matrix C. Then

$$B = BI = B(AC) = (BA)C = DC.$$

Since D has a zero row, DC also has a zero row, and so B has a zero row. But B is invertible, and it is impossible for an invertible matrix to have a zero row. So there can be no such matrix C satisfying $AC = I$.

6. Find the general solution of the matrix equation $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = X \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution.

Let $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x+2z & y+2w \\ 3x+4z & 3y+4w \end{bmatrix}$$

and

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} x+3y & 2x+4y \\ z+3w & 2z+4w \end{bmatrix},$$

giving us a system of four linear equations equivalent to the original matrix equation:

$$\begin{aligned} x+2z &= x+3y \\ y+2w &= 2x+4y \\ 3x+4z &= z+3w \\ 3y+4w &= 2z+4w. \end{aligned}$$

The first equation and the last both say $2z = 3y$; so it will do no harm to omit the last equation. We are left with three equations in the four unknowns x, y, z and w, and we find that the corresponding augmented matrix is as follows:

$$\left[\begin{array}{cccc|c} 0 & -3 & 2 & 0 & 0 \\ -2 & -3 & 0 & 2 & 0 \\ 3 & 0 & 3 & -3 & 0 \end{array}\right].$$

Perform the following elementary row operations: multiply the 3rd row by $1/3$, then swap the 1st and 3rd rows, then add twice the 1st row to the 2nd . This gives

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & -3 & 2 & 0 & 0 \\ 0 & -3 & 2 & 0 & 0 \end{array}\right].$$

Now subtract the 2nd row from the 3rd, and then multiply the 2nd row by $-1/3$, and we obtain the following matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This is a reduced row-echelon matrix. The leading variables are x and y, while z and w are free. If we let $w = u$ and $z = 3t$ (where u and t are arbitrary parameters) then we get $y = 2t$ and $x = -3t + u$. So the general solution of the original matrix equation is

$$X = \begin{bmatrix} -3t+u & 2t \\ 3t & u \end{bmatrix} = t \begin{bmatrix} -3 & 2 \\ 3 & 0 \end{bmatrix} + u \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where t and u are arbitrary. As a matter of interest, observe that since

$$\begin{bmatrix} -3 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the general solution can also be written as

$$X = \alpha \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \beta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where α and β are arbitrary.

7. Let A be an $r \times n$ matrix. For each value of i from 1 to n, let $\mathbf{e}_k$ be the k-th column of the $n \times n$ identity matrix.

(i) Using the Kronecker delta, write down the formula for the j-th entry of $\mathbf{e}_k$ (where j and k are arbitrary numbers in $\{1, 2, \ldots, n\}$).

(ii) Using sigma notation, write down the expression for the i-th entry of the column vector $A\mathbf{e}_k$ (for arbitrary i in $\{1, 2, \ldots, r\}$ and k in $\{1, 2, \ldots, n\}$), and hence show that $A\mathbf{e}_k$ is the k-th column of A.

Solution.

(i) The k-th column of the identity matrix has a 1 in the the k-th position and 0's elsewhere. That is to say, the j-th entry of the k-th column is 1 if $j = k$ and 0 if $j \neq k$. Thus the j-th entry of $\mathbf{e}_k$ is δ_{jk}, for all j, $k \in \{1, 2, \ldots, n\}$.

(ii) The i-th entry of $A\mathbf{e}_k$ is obtained by multiplying the i-th row of A by $\mathbf{e}_k$. That is, get the product of a_{i1} and the first entry of $\mathbf{e}_k$, add this to the product of a_{i2} and the second entry of $\mathbf{e}_k$, add this to the product of a_{i2} and the second entry of $\mathbf{e}_k$, and so on, the last term added on being the product of a_{in} and the last entry of $\mathbf{e}_k$. In sigma notation, this is $\sum_{j=1}^{n} a_{ij}\delta_{jk}$ (since δ_{jk} is the j-th entry of $\mathbf{e}_k$).

Now since δ_{jk} is zero when $j \neq k$, the terms in the sum $\sum_{j=1}^{n} a_{ij}\delta_{jk}$ for which $j \neq k$ are zero, and hence $\sum_{j=1}^{n} a_{ij}\delta_{jk} = a_{ik}\delta_{kk} = a_{ik}$. But this is the i-th entry of the k-th column of A. So we have shown that the i-th entry of $A\mathbf{e}_k$ is the same as the i-th entry of the k-th column of A—and this ture for all values of i. So $A\mathbf{e}_k$ is equal to the k-th column of A.

8. Use row operations to calculate the determinant of the matrix

$$\begin{bmatrix} x & 0 & 0 & 0 & 0 & a_0 \\ -1 & x & 0 & 0 & 0 & a_1 \\ 0 & -1 & x & 0 & 0 & a_2 \\ 0 & 0 & -1 & x & 0 & a_3 \\ 0 & 0 & 0 & -1 & x & a_4 \\ 0 & 0 & 0 & 0 & -1 & x + a_5 \end{bmatrix}$$

Solution.

Multiplying the second row by x and then adding the first row to the second row changes the second row to

$$[\,0 \quad x^2 \quad 0 \quad 0 \quad 0 \quad a_1x + a_0\,].$$

and multiplies the determinant by x.

Multiplying the third row by x^2 and adding the second row to the third row changes the third row to

$$[\,0 \quad 0 \quad x^3 \quad 0 \quad 0 \quad a_2x^2 + a_1x + a_0\,].$$

and multiplies the determinant by x^2.

Multiplying the fourth row by x^3 and adding the third row to the fourth row gives

$$[\,0 \quad 0 \quad 0 \quad x^4 \quad 0 \quad a_3x^3 + a_2x^2 + a_1x + a_0\,].$$

Multiply the fifth row by x^4 and add the fourth row to it, then multiply the sixth row by x^5 and add the fifth row to it. We have produced an upper triangular matrix with diagonal entries x, x^2, x^3, x^4, x^5 and $x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and in doing so have multiplied the determinant by $x\,x^2\,x^3\,x^4\,x^5$. Hence the determinant of the original matrix is

$$x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Answers to Preparatory Questions

1. $\begin{bmatrix} 2 & 1 & 2 \\ 3 & -4 & 4 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+1+4 \\ 3-4+8 \\ 1+3+10 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. So the given vector is an eigenvector with eigenvalue 7.

2. Apply row operations to A reduce it to I:

$$\begin{bmatrix} 1 & 1 & 5 \\ 2 & 2 & -3 \\ 2 & 4 & 10 \end{bmatrix} \xrightarrow[R_3:=R_3-2R_1]{R_2:=R_2-2R_1} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & -13 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & -13 \end{bmatrix}$$

$$\xrightarrow[R_3:=-\frac{1}{13}R_3]{R_2:=\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1:=R_1-R_2]{R_1:=R_1-5R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now A is the product of the elementary matrices corresponding to the inverses of the elementary row operations used, in the order they were used. So

$$A = \begin{bmatrix} 1&0&0 \\ 2&1&0 \\ 0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 2&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{bmatrix}\begin{bmatrix} 1&0&0 \\ 0&2&0 \\ 0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&-13 \end{bmatrix}\begin{bmatrix} 1&0&5 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}\begin{bmatrix} 1&1&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}.$$

The determinants of these elementary matrices are (respectively) 1, 1, -1, 2, -13, 1 and 1. And so $\det A = (-1) \times 2 \times (-13) = 26$.

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 11.