The University of Sydney

MATH1902 Linear Algebra (advanced)

Exercises for Week 12 (beginning May 25) Semester 1 2009

Preliminary Reading:

Chapter 3 of the Linear Algebra book.

Objectives:

By the end of Week 12, to achieve at least a pass level, you should be able to

12A: calculate the characteristic equation of a matrix,

12B: calculate the eigenvalues and eigenvectors of 3×3 matrices.

To achieve higher than a pass level you should be able to

12C: work with matrix equations involving inverses and adjoints,

12D: carry out symbolic calculations with eigenvalues and eigenvectors.

Preparatory questions. (Answers are on the next page.)

1. For which values of λ is the following matrix not invertible:

$$\begin{bmatrix} 3-\lambda & -2 \\ -1 & 2-\lambda \end{bmatrix}.$$

2. Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$.

3. For each eigenvalue of the matrix A of the previous question, find a corresponding eigenvector.

Practice questions

4. Find the eigenvalues and corresponding eigenvectors for $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 1 & 13 \\ 0 & 0 & -3 \end{bmatrix}$.

Solution.

First calculate the eigenvalues of A. By expanding the determinant of $\det(A-\lambda I)$ along its last row we see that the characteristic equation is

$$\begin{aligned} 0 &= \det \begin{bmatrix} 1-\lambda & -1 & 5 \\ -1 & 1-\lambda & 13 \\ 0 & 0 & -3-\lambda \end{bmatrix} \\ &= (-3-\lambda)((1-\lambda)^2-1) = -(3+\lambda)\lambda(\lambda-2). \end{aligned}$$

This has roots $\lambda = -3$, 0, 2 and these are the eigenvalues of A.

Any nonzero solution of $(A-\lambda I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector for the eigenvalue λ. Thus to find the eigenvectors for the eigenvalue 2 we should find the nonzero solutions of

$$\begin{bmatrix} -1 & -1 & 5 \\ -1 & -1 & 13 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is easily checked that the general solution of this system is $x = -y = t$ and $z = 0$, where t is an arbitrary parameter. Thus the eigenvectors of A for the eigenvalue 2 are all column vectors of the form $t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ for $t \neq 0$. Similarly, the eigenvectors for the eigenvalue -3 are the nonzero scalar multiples of $\begin{bmatrix} 11 \\ 19 \\ -5 \end{bmatrix}$, and those for the eigenvalue 0 are the nonzero scalar multiples of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

5. [The Cayley-Hamilton Theorem for 2×2 matrices.] Show that the characteristic equation of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$. Show also that A satisfies the matrix equation $A^2 - (a+d)A + (ad-bc)I_2 = 0_2$, where I_2 and 0_2 are the 2×2 identity and zero matrices respectively.

Solution.

$$\det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc).$$

Since $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix}$ we find that $A^2 - (a+d)A + (ad-bc)I_2$ equals

$$\begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cb+d^2 \end{bmatrix} - \begin{bmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

as required. (The Cayley-Hamilton Theorem states that a square matrix satisfies its characteristic equation. We have proved it for 2×2 matrices.)

6. Let A and P be $n \times n$ matrices, with P invertible. Show that A and PAP^{-1} have the same characteristic equation. (Use the rule that $\det(XY) = (\det X)(\det Y)$. And note that $\det X \det Y$ always equals $\det Y \det X$, even though XY need not equal YX.)

Solution.

Note that $\det P \det(P^{-1}) = \det(PP^{-1}) = \det I = 1$. Note also that

$$P(\lambda I)P^{-1} = \lambda(PI)P{-1} = \lambda I.$$

It follows that

$$\det(PAP^{-1} - \lambda I) = \det(PAP^{-1} - P\lambda IP^{-1}) = \det(P(AP^{-1} - \lambda IP^{-1}))$$
$$= \det(P(A - \lambda I)P^{-1}) = \det P \det(A - \lambda I)(\det P)^{-1} = \det(A - \lambda I).$$

7. (*i*) Given a 3×3 matrix A and a 3×1 column vector $\mathbf{b}$, show that

$$(\operatorname{adj} A)\mathbf{b} = \begin{bmatrix} \det(A_1) \\ \det(A_2) \\ \det(A_3) \end{bmatrix}$$

where A_i is the matrix obtained from A by replacing column i by $\mathbf{b}$.

(*ii*) Suppose that A is invertible and then show that the solution to the matrix equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \det(A_1)/\det(A) \\ \det(A_2)/\det(A) \\ \det(A_3)/\det(A) \end{bmatrix}.$$

[This formula for the solution of a system of n equations in n unknowns is known as *Cramer's Rule*. (We have only presented the case $n = 3$ here.) Note that the coefficient matrix must be invertible for Cramer's Rule to apply.]

(*iii*) Use (*ii*) to solve the following equations:

$$\begin{aligned} x + 2y + 2z &= 5 \\ x + 3y + z &= 0 \\ x + 3y + 2z &= -2 \end{aligned}$$

Which method of solving equations do you prefer: using row operations or Cramer's rule?

Solution.

(*i*) The entries of the first row of $\operatorname{adj} A$ are the cofactors c_{i1} of the elements of the first column of A. Thus the matrix product of the first row of $\operatorname{adj} A$ and $\mathbf{b}$ is $\sum_{i=1}^{3} c_{i,1} b_i$, where b_i is the entry in row i of $\mathbf{b}$. This is just the expansion down the first column of the matrix A_1 obtained from A by replacing its first column with $\mathbf{b}$. Thus its value is $\det(A_1)$. The same argument shows that the entries in the second and third rows of $(\operatorname{adj} A)\mathbf{b}$ are $\det(A_2)$ and $\det(A_3)$.

(*ii*) Since A is invertible we may multiply the equation $A\mathbf{x} = \mathbf{b}$ on the left by A^{-1} to obtain $\mathbf{x} = A^{-1}\mathbf{b}$. But we know that $A^{-1} = (\det A)^{-1} \operatorname{adj} A$ and so $\mathbf{x} = (\det A)^{-1}(\operatorname{adj} A)\mathbf{b}$. The result now follows from (*i*).

(*iii*) The matrix of coefficients is

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}.$$

Expanding across the first row, the determinant of A is

$$(3 \times 2 - 1 \times 3) - 2(1 \times 2 - 1 \times 1) + 2(1 \times 3 - 3 \times 1) = 3 - 2 = 1.$$

Furthermore, $\det(A_1) = 23$, $\det(A_2) = -7$ and $\det(A_3) = -2$. Therefore, $x = 23$, $y = -7$ and $z = -2$.

8. The *Hessian* of a function $u(x_1, x_2)$ of two variables is the determinant of the 2×2 matrix whose (i, j)-th entry is $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$. Find the Hessian of $ax_1^2 + bx_1x_2 + cx_2^2$.

Solution.

If $u = ax_1^2 + bx_1x_2 + cx_2^2$, then

$$\frac{\partial u}{\partial x_1} = 2ax_1 + bx_2 \qquad \frac{\partial^2 u}{\partial x_1 \partial x_1} = 2a \qquad \frac{\partial^2 u}{\partial x_1 \partial x_2} = b$$
$$\frac{\partial u}{\partial x_2} = bx_1 + 2cx_2 \qquad \frac{\partial^2 u}{\partial x_2 \partial x_2} = 2c$$

and so the Hessian is $\begin{vmatrix} 2a & b \\ b & 2c \end{vmatrix} = 4ac - b^2$.

Answers to Preparatory Questions

1. The determinant of the given matrix is $(\lambda - 1)(\lambda - 4)$. This is 0 when $\lambda = 1$ or $\lambda = 4$ and so these are the values for which the matrix is not invertible.

2. The calculation of the preceding exercise shows that the eigenvalues of A are 1 and 4.

3. When $\lambda = 1$ an eigenvector is any non-zero multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. When $\lambda = 4$ an eigenvector is any non-zero multiple of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

`www.maths.usyd.edu.au/u/UG/JM/MATH1902/`

and then do the Web Quiz for Week 12.