

Preliminary Reading:

Chapter 3 of the Linear Algebra book.

Objectives:

By the end of Week 13, to achieve at least a pass level, you should be able to

13A: calculate the characteristic equation of a matrix,

13B: calculate the eigenvalues and eigenvectors of 3×3 matrices.

To achieve higher than a pass level you should be able to

13C: diagonalize a matrix,

13D: determine when column vectors are linearly independent.

Preparatory questions. (Answers are on the next page.)

1. Let $A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & -2 & 2 \\ 3 & -6 & 5 \end{bmatrix}$.

(i) Find the characteristic polynomial of A .

(ii) Find the determinant of A .

(iii) The sum of the diagonal elements of A is one of the coefficients of the characteristic polynomial. Which one?

2. Let A be an $n \times n$ matrix and B an $n \times r$ matrix. Suppose that $AB = \lambda B$ for some scalar λ . Show that each column of B is either zero or an eigenvector of A .

3. Suppose that A is a 3×3 matrix such that the sum of the columns of A is $\mathbf{0}$. Write down an eigenvector and the corresponding eigenvalue for A .

Practice questions

4. In each case find the eigenvalues and corresponding eigenvectors for the matrix A , and hence find a matrix M such that $M^{-1}AM$ is diagonal.

(i) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Solution.

(i) The characteristic equation is

$$0 = \begin{vmatrix} 1 - \lambda & 2 \\ -1 & 4 - \lambda \end{vmatrix} = (1 - \lambda)(4 - \lambda) + 2 = \lambda^2 - 5\lambda + 6.$$

So the eigenvalues are 2 and 3.

To find an eigenvector with eigenvalue 2, solve $\begin{bmatrix} 1-2 & 2 \\ -1 & 4-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. One solution is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Similarly we find that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue 3. The matrix M that has these two eigenvectors as its columns then has the property that $M^{-1}AM$ is diagonal, the diagonal entries being the eigenvalues:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

(It is a theorem that if the columns of a square matrix M are eigenvectors corresponding to distinct eigenvalues of some matrix, then M is necessarily invertible. It is easy to verify this result in this example by showing that $\det M \neq 0$.)

(ii) The characteristic equation is

$$(3 - \lambda)((-1 - \lambda)(-\lambda) - 1) - 2(2\lambda - 1) + (-2 - (-1 - \lambda)) = 0.$$

Simplifying the polynomial, we find that this becomes $-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$. The left hand side now factorises as $(\lambda - 2)(1 - \lambda)(1 + \lambda)$, and so the eigenvalues are 2, 1 and -1 . To find corresponding eigenvectors we must find one nonzero solution for each of the following systems of equations:

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 & 1 \\ -2 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 4 & 2 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Suitable solutions are

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix} \quad \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

and we deduce that $M = \begin{bmatrix} -5 & 1 & 1 \\ 3 & -1 & -3 \\ -1 & 0 & 2 \end{bmatrix}$ satisfies

$$M^{-1}AM = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

5. (i) Show that the characteristic polynomials of the matrices $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ are both equal to $(x - 1)^3(x - 5)$.

(ii) Find an invertible matrix T such that $T^{-1}AT$ is diagonal.

(iii) Show that there is no invertible matrix U such that $U^{-1}BU$ is diagonal.

Solution.

(i) Observe that $\det(A - xI)$ is

$$\begin{vmatrix} 2-x & 1 & 1 & 1 \\ 1 & 2-x & 1 & 1 \\ 1 & 1 & 2-x & 1 \\ 1 & 1 & 1 & 2-x \end{vmatrix} = \begin{vmatrix} 1-x & -1+x & 0 & 0 \\ 1 & 2-x & 1 & 1 \\ 1 & 1 & 2-x & 1 \\ 0 & 0 & -1+x & 1-x \end{vmatrix}$$

since $R1 := R1 - R2$ and $R4 := R4 - R3$ do not change the determinant. Now taking out a common factor of $(1-x)$ from the first row and also from the last row shows that

$$\begin{vmatrix} 1-x & -1+x & 0 & 0 \\ 1 & 2-x & 1 & 1 \\ 1 & 1 & 2-x & 1 \\ 0 & 0 & -1+x & 1-x \end{vmatrix} = (1-x)^2 \begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 2-x & 1 & 1 \\ 1 & 1 & 2-x & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix}.$$

Applying now $R2 := R2 - R1$ and $R3 := R3 - R1$ shows that this equals

$$(1-x)^2 \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 3-x & 1 & 1 \\ 0 & 2 & 2-x & 1 \\ 0 & 0 & -1 & 1 \end{vmatrix} = (1-x)^2 \begin{vmatrix} 3-x & 1 & 1 \\ 2 & 2-x & 1 \\ 0 & -1 & 1 \end{vmatrix},$$

and we can obtain another factor of $1-x$ by subtracting Row 2 from Row 1:

$$\begin{aligned} (1-x)^2 \begin{vmatrix} 3-x & 1 & 1 \\ 2 & 2-x & 1 \\ 0 & -1 & 1 \end{vmatrix} &= (1-x)^2 \begin{vmatrix} 1-x & -1+x & 0 \\ 2 & 2-x & 1 \\ 0 & -1 & 1 \end{vmatrix} \\ &= (1-x)^3 \begin{vmatrix} 1 & -1 & 0 \\ 2 & 2-x & 1 \\ 0 & -1 & 1 \end{vmatrix}. \end{aligned}$$

Finally, applying $R2 := R2 - 2R1$ we deduce that $\det(A - xI)$ equals

$$(1-x)^3 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 4-x & 1 \\ 0 & -1 & 1 \end{vmatrix} = (1-x)^3 \begin{vmatrix} 4-x & 1 \\ -1 & 1 \end{vmatrix} = (1-x)^3(5-x).$$

The calculation for the other matrix, B , is much more straightforward. Since B is upper triangular, we see directly that $\det(B - xI) = (1-x)^3(5-x)$.

(ii) To find 1-eigenvectors for A , solve $(A - I)\mathbf{x} = \mathbf{0}$, that is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly the reduced echelon matrix for this system (obtained by subtracting the first row from each of the others) has just one nonzero row. The variables y , z and w are free, and x is the only leading variable. The general solution is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -r - s - t \\ r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where r , s and t are arbitrary parameters. First putting $r = 1, s = 0, t = 0$, then $r = 0, s = 1, t = 0$ and then $r = 0, s = 0, t = 1$ yields the three eigenvectors

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

It is easily seen that the only solution to $r\mathbf{x}_1 + s\mathbf{x}_2 + t\mathbf{x}_3 = \mathbf{0}$ is $r = s = t = 0$; that is, \mathbf{x}_1 , \mathbf{x}_2 and \mathbf{x}_3 are linearly independent.

To find a 5-eigenvector for A , solve $(A - 5I)\mathbf{x} = \mathbf{0}$. It is a routine matter to show that the general solution is

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Now let T be a matrix whose columns are our three linearly independent 1-eigenvectors and a 5-eigenvector. Any matrix constructed in this way will necessarily be invertible (although the theory that guarantees this has not been covered in this course). A suitable such matrix is

$$T = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

and we see that

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Thus $T^{-1}AT = D$, where D is the diagonal matrix (above) with diagonal entries 1, 1, 1 and 5.

(iii) Suppose that U is an invertible matrix such that $U^{-1}BU = D'$, a diagonal matrix. We shall show that this leads to a contradiction. We have $BU = UD'$. Now if the columns of U are \mathbf{u}_1 , \mathbf{u}_2 , \mathbf{u}_3 and \mathbf{u}_4 , and the diagonal entries of D' are λ_1 , λ_2 , λ_3 and λ_4 , then the columns of UD' are $\lambda_1\mathbf{u}_1$, $\lambda_2\mathbf{u}_2$, $\lambda_3\mathbf{u}_3$ and $\lambda_4\mathbf{u}_4$. Furthermore, the columns of BU are $B\mathbf{u}_1$, $B\mathbf{u}_2$, $B\mathbf{u}_3$ and $B\mathbf{u}_4$. In other words, the columns of U must be eigenvectors of B . (Note that the columns of U must all be nonzero since U is invertible.)

The eigenvalues of B are 1 and 5. A short calculation shows that the general solutions of $(B - I)\mathbf{x} = \mathbf{0}$ and $(B - 5I)\mathbf{y} = \mathbf{0}$ are (respectively)

$$\mathbf{x} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y} = \beta \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where α and β are arbitrary parameters. But since the four columns of U are all meant to be eigenvectors of B , at least one of these columns must be a scalar multiple of another. This forces the determinant of U to be zero, which is incompatible with the fact that U is invertible. (We have seen in lectures that a matrix with two equal rows must have zero determinant, and it follows readily that if one row is a scalar multiple of another then the matrix has zero determinant. The same applies for columns, as follows from that fact that transposing a matrix does not change the determinant). In this particular example there is a slightly easier argument: whichever eigenvectors of B are chosen as the columns of U , the second row of U will be zero. So there is no invertible U whose columns are eigenvectors of B ; so B cannot be diagonalized.

6. Let A be an $n \times n$ matrix and \mathbf{u} a $1 \times n$ matrix. (That is, \mathbf{u} is a row vector.) Show that if $\mathbf{u}A = \lambda\mathbf{u}$ for some scalar λ then either \mathbf{u} is zero or λ is a root of the characteristic polynomial of A .

Solution.

Rearranging $\mathbf{u}A = \lambda\mathbf{u}$ we find that $\mathbf{u}(A - \lambda I) = \mathbf{0}$. If $(A - \lambda I)$ is invertible then we obtain

$$\mathbf{u}(A - \lambda I)(A - \lambda I)^{-1} = \mathbf{0}(A - \lambda I)^{-1} = \mathbf{0},$$

and therefore $\mathbf{u} = \mathbf{0}$. So if $\mathbf{u} \neq \mathbf{0}$ then $A - \lambda I$ is not invertible, which means that $\det(A - \lambda I) = 0$. This in turn means that λ is a root of the characteristic polynomial.

7. Let A be a 3×3 matrix. Show that in the first row expansion of $\det(A - xI)$, only the first term involves x^2 and x^3 , and hence check that the coefficient of x^2 is $a_{11} + a_{22} + a_{33}$. Similarly check that when A is 4×4 the coefficient of x^3 is $-\sum_{i=1}^4 a_{ii}$. Generalise this to the $n \times n$ case. (The sum of the entries on the main diagonal of a square matrix is called the *trace* of the matrix. It equals the sum of the eigenvalues, counted according to their multiplicities as roots of the characteristic polynomial.)

Solution.

By the first row expansion, $\det(A - xI)$ equals

$$(a_{11} - x) \begin{vmatrix} a_{22} - x & a_{23} \\ a_{32} & a_{33} - x \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} - x \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} - x \\ a_{31} & a_{32} \end{vmatrix}.$$

Since x appears only once in each of the second and third terms here it is clear that we will never get x^2 or x^3 arising from these terms when they are expanded. And indeed by expanding the first term we see that the coefficients of x^2 and x^3 in this are the same as in $(a_{11} - x)(a_{22} - x)(a_{33} - x)$. When this is expanded there are three terms which are scalar multiples of x^2 , the coefficients being a_{11} , a_{22} and a_{33} respectively.

Given an $n \times n$ matrix A with entries a_{ij} and characteristic polynomial $p(x) = \det(A - xI_n)$ we shall show that the coefficient of x^n in $p(x)$ is $(-1)^n$ and the coefficient of x^{n-1} in $p(x)$ is $(-1)^{n-1}(a_{11} + a_{22} + \dots + a_{nn})$. We do this by induction on n . We have already checked that the statement is true for $n = 3$ and it is easy to check it directly for $n = 2$. So we suppose that it is true for $n - 1$ and prove that it holds for n .

Let A_{11} be the minor obtained by deleting the first row and column of A . Expanding $\det(A - xI_n)$ across the first row gives

$$\det(A - xI_n) = (a_{11} - x) \det(A_{11} - xI_{n-1}) + \text{other terms.}$$

The highest power of x in the *other terms* is x^{n-2} because they all come from determinants obtained from $A - xI_n$ by crossing out a row containing an x and a column containing an x . This means that we may safely ignore the *other terms*. By induction the coefficient of x^{n-1} in $\det(A_{11} - xI_{n-1})$ is $(-1)^{n-1}$ and the coefficient of x^{n-2} is $(-1)^{n-2}(a_{22} + a_{33} + \cdots + a_{nn})$. Thus the part of $p(x)$ that contains the x^n and x^{n-1} terms is

$$\begin{aligned} (a_{11} - x)((-1)^{n-1}x^{n-1} + (-1)^{n-2}(a_{22} + a_{33} + \cdots + a_{nn})x^{n-2}) \\ = (-1)^n x^n + (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})x^{n-1} + \cdots \end{aligned}$$

That is, the coefficient of x^{n-1} in $p(x)$ is $(-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})$ and the coefficient of x^n is $(-1)^n$.

8. Let $M = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$. Show that each of the columns of M is an eigenvector for the matrix $A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$. By considering its determinant, show that M is invertible. Without calculating M^{-1} , show that $M^{-1}AM$ is a diagonal matrix, and evaluate it.

Solution.

$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 + 3 + 3 \\ 0 + 2 - 3 \\ 0 + 0 + 3 \end{bmatrix} = (-1) \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}$, which shows that the last column of M is an eigenvector of A for the eigenvalue -1 . Similarly,

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

show that the second column of M is an eigenvector of A for the eigenvalue 2, and the first column of M is an eigenvector for A with eigenvalue 1. Combining the above equations into a single equation we obtain

$$\begin{aligned} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix} &= \begin{bmatrix} 1 & 6 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \end{aligned}$$

or $AM = MD$, where D is the diagonal matrix whose diagonal entries are the three eigenvalues of A corresponding to the eigenvectors which are the columns of M (in the same order). Since M is upper triangular its determinant is the product of its diagonal entries, which is -3 . So $\det M \neq 0$, and hence M^{-1} exists. Now the equation $AM = MD$ yields

$$M^{-1}AM = M^{-1}MD = ID = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

9. Let \mathbf{u} be a nonzero n -component row vector and \mathbf{v} a nonzero n -component column vector, and let A be the $n \times n$ matrix $\mathbf{v}\mathbf{u}$.

- (i) Show that \mathbf{v} is an eigenvector for A corresponding to the eigenvalue $\mathbf{u}\mathbf{v}$.
- (ii) Show that every nonzero n -component column vector \mathbf{w} such that $\mathbf{u}\mathbf{w} = 0$ is an eigenvector for A with eigenvalue 0.
- (iii) By taking $\mathbf{u} = [1 \ 2 \ -1]$ and $\mathbf{v} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$, use the results of Parts (i) and (ii) to

find a (-1) -eigenvector and two 0-eigenvectors for the matrix $\begin{bmatrix} 4 & 8 & -4 \\ -1 & -2 & 1 \\ 3 & 6 & -3 \end{bmatrix}$, such that the 0-eigenvectors are not scalar multiples of each other.

Hence find an invertible matrix M such that $AM = M \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution.

(i) Note that \mathbf{v} is an $n \times 1$ matrix and \mathbf{u} a $1 \times n$ matrix; so it is indeed true that $\mathbf{v}\mathbf{u}$ is an $n \times n$ matrix. The product $\mathbf{u}\mathbf{v}$ is also defined, and is a 1×1 matrix; that is, $\lambda = \mathbf{u}\mathbf{v}$ is a scalar. Now we have

$$A\mathbf{v} = (\mathbf{v}\mathbf{u})\mathbf{v} = \mathbf{v}(\mathbf{u}\mathbf{v}) = \mathbf{v}\lambda,$$

and since $\mathbf{v} \neq \mathbf{0}$ this shows that \mathbf{v} is an eigenvector for A for the eigenvalue λ , as required.

(ii) If $\mathbf{u}\mathbf{w} = 0$ then $A\mathbf{w} = (\mathbf{v}\mathbf{u})\mathbf{w} = \mathbf{v}(\mathbf{u}\mathbf{w}) = \mathbf{0} = 0\mathbf{w}$, and if also $\mathbf{w} \neq \mathbf{0}$ this shows that \mathbf{w} is a 0-eigenvector for A .

(iii) For the given values of \mathbf{u} and \mathbf{v} we find that $A = \mathbf{v}\mathbf{u}$ is $\begin{bmatrix} 4 & 8 & -4 \\ -1 & -2 & 1 \\ 3 & 6 & -3 \end{bmatrix}$.

We also find that $\mathbf{u}\mathbf{v} = 4 - 2 - 3 = -1$. So \mathbf{v} is a -1 -eigenvector for A , and since it is also readily checked that the vectors $\mathbf{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ satisfy $\mathbf{u}\mathbf{w}_1 = \mathbf{u}\mathbf{w}_2 = 0$ we conclude that \mathbf{w}_1 and \mathbf{w}_2 are 0-eigenvectors for A . If we now take M to be the matrix whose three columns are \mathbf{v} , \mathbf{w}_1 and \mathbf{w}_2 ,

$$M = \begin{bmatrix} 4 & 2 & 1 \\ -1 & -1 & 0 \\ 3 & 0 & 1 \end{bmatrix},$$

then it follows that $AM = M \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. It remains to check that M is invertible, which can be done by calculating $\det M$. Expanding along the second row we find that $\det M = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = 2 - (4 - 3) = 1 \neq 0$; hence M is invertible.

Answers to Preparatory Questions

1. (i) We have

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3 - \lambda & -2 & 1 \\ 2 & -2 - \lambda & 2 \\ 3 & -6 & 5 - \lambda \end{vmatrix} \\ &= (3 - \lambda) \begin{vmatrix} -2 - \lambda & 2 \\ -6 & 5 - \lambda \end{vmatrix} + 2 \begin{vmatrix} 2 & 2 \\ 3 & 5 - \lambda \end{vmatrix} + \begin{vmatrix} 2 & -2 - \lambda \\ 3 & -6 \end{vmatrix} \\ &= (3 - \lambda)((-2 - \lambda)(5 - \lambda) + 12) + 2(10 - 2\lambda - 6) + (-12 + 6 + 3\lambda) \\ &= 8 - 12\lambda + 6\lambda^2 - \lambda^3 \\ &= -(\lambda - 2)^3 \end{aligned}$$

(ii) The determinant is the value of the characteristic polynomial when $\lambda = 0$, namely 8.

(iii) The sum of the diagonal elements is 6, the coefficient of λ^2 . (It is always true that for an $n \times n$ matrix the sum of the diagonal elements is the coefficient of $(-\lambda)^{n-1}$ in the characteristic polynomial. See exercise 7.)

2. $AB = \lambda B$ gives $A\mathbf{b} = \lambda\mathbf{b}$ for each column \mathbf{b} of the matrix B . If \mathbf{b} is nonzero then by definition this column is a λ -eigenvector of A .
3. To say that the sum of the columns of A is $\mathbf{0}$ is to say that their linear combination with coefficients all 1 is $\mathbf{0}$. That is $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and so $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 0.

Web Quiz

There are additional self assessment tasks on the Web. Go to the Web page at

www.maths.usyd.edu.au/u/UG/JM/MATH1902/

and then do the Web Quiz for Week 13.