1. (10 marks). Let π be the plane given by the equation $3x - y - 2z = -3$.

(a) Find parametric scalar equations for the line ℓ which passes through the point $A(1, 0, -4)$ and which is perpendicular to π.

(b) Find the coordinates of the intersection point B of ℓ and π.

(c) Hence calculate the distance from A to π.

(d) Find the Cartesian equation of the plane π' through the point $A(1, 0, -4)$ which is parallel to the plane π.

(e) Find all values of c for which the plane $3x - y + c\,z = -3$ is perpendicular to the plane π.

Solution

(a) The line ℓ is parallel to the vector $3\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}$. Hence, the equations are $x = 1 + 3t, \quad y = -t, \quad z = -4 - 2t$.

(b) Find the value of t from $3(1 + 3t) - (-t) - 2(-4 - 2t) = -3$. This gives $t = -1$ and so $B(-2, 1, -2)$.

(c) The distance is $AB = \sqrt{3^2 + (-1)^2 + (-2)^2} = \sqrt{14}$.

(d) The equation is $3(x - 1) - (y - 0) - 2(z + 4) = 0$, that is,
$3x - y - 2z = 11$.

(e) The condition means that $3^2 + (-1)^2 - 2c = 0$ and so $c = 5$.

2. (a) **(6 marks)**. The line m is given by the equations $\dfrac{x}{2} = \dfrac{y+7}{3} = \dfrac{z-6}{-11}$ and the plane ρ is given by the equation $5x - 2y + z = 9$.

(*i*) Show that m and ρ are not parallel.

(*ii*) Use vector product to find a nonzero vector which is perpendicular to m and parallel to ρ.

(*iii*) Find the Cartesian equation of a plane which is perpendicular to the plane ρ and contains the line m.

(b) **(4 marks)**.

(*i*) Given that the volume of a pyramid is found as one third of the product of the area of the base and the height, show that the volume of a tetrahedron $ABCD$ can be given by $V = \frac{1}{6}\,\big|\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})\big|$.

(*ii*) Calculate the volume of the tetrahedron $ABCD$ with $A(1,2,3)$, $B(-1,0,5)$, $C(0,3,1)$ and $D(2,2,2)$.

Solution

(a) (*i*) The line m is parallel to the vector $\mathbf{u} = 2\,\mathbf{i} + 3\,\mathbf{j} - 11\,\mathbf{k}$ while ρ is perpendicular to $\mathbf{v} = 5\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$. Since $\mathbf{u}\cdot\mathbf{v} = 10 - 6 - 11 = -7 \neq 0$, the line m is not parallel to ρ.

(*ii*) By the properties of the vector product, the vector $\mathbf{n} = \mathbf{u}\times\mathbf{v}$ is perpendicular to m and parallel to ρ. Its coordinates are found by

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -11 \\ 5 & -2 & 1 \end{vmatrix} = -19\,\mathbf{i} - 57\,\mathbf{j} - 19\,\mathbf{k}.$$

Thus, the required vector is $\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$.

(*iii*) By the previous part, the vector $\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$ is normal to the required plane. This point $(0,-7,6)$ belongs to the plane and so the equation is $x + 3(y+7) + (z-6) = 0$, that is, $x + 3y + z = -15$.

(b) (*i*) The area of the base ABC can be found as the half of the magnitude of the cross product $\overrightarrow{AB}\times\overrightarrow{AC}$. Hence, the volume equals $\frac{1}{6}\,|\overrightarrow{AD}|\,|\overrightarrow{AB}\times\overrightarrow{AC}|\,|\cos\theta|$, where θ is the angle between the vectors $\overrightarrow{AD}$ and $\overrightarrow{AB}\times\overrightarrow{AC}$. Hence, by the definition of the dot product, $V = \frac{1}{6}\,\big|\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})\big|$.

(*ii*) We have $\overrightarrow{AB} = -2\,\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}$, $\overrightarrow{AC} = -\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$, $\overrightarrow{AD} = \mathbf{i} - \mathbf{k}$. Hence, using the formula from lectures for the triple scalar product, we have

$$\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC}) = \begin{vmatrix} 1 & 0 & -1 \\ -2 & -2 & 2 \\ -1 & 1 & -2 \end{vmatrix} = 6.$$

Thus, $V = 1$.

3. (10 marks). For the system of linear equations

$$\begin{cases} x_1 + x_2 + 2x_3 + x_4 = 2 \\ 2x_1 + x_2 + 2x_3 + x_4 = 2 \\ 2x_1 + x_2 + 2x_3 + 2x_4 = 4 \\ 3x_1 + 2x_2 + 4x_3 + 2x_4 = 4 \end{cases}$$

(a) Write down the augmented coefficient matrix.

(b) Use elementary row operations to bring the augmented coefficient matrix into the reduced row echelon form.

(c) Write down the general solution of the system.

(d) Find all values of the parameters a, b, c and d such that every solution of the above system is also a solution of another system of linear equations given by

$$\begin{cases} a\,x_1 + b\,x_2 + c\,x_3 + d\,x_4 = 0 \\ b\,x_1 + a\,x_2 + c\,x_3 = 0 \\ d\,x_3 + a\,x_4 = 2. \end{cases}$$

Solution

(a) The augmented coefficient matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 2 & 4 \\ 3 & 2 & 4 & 2 & 4 \end{array}\right].$$

(b) Using Gaussian elimination and starting with the elementary row operations $R_2 := R_2 - 2R_1$, $R_3 := R_3 - 2R_1$, $R_4 := R_4 - 3R_1$, we get the matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 0 & -1 & -2 & -1 & -2 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & -1 & -2 & -1 & -2 \end{array}\right].$$

Now apply $R_2 := -R_2$ followed by $R_3 := R_3 + R_2$ and $R_4 := R_4 + R_2$ to get

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This matrix is in row echelon form. In order to bring it into the reduced row echelon form, omit the row of zeros and apply $R_1 := R_1 - R_3$ and $R_2 := R_2 - R_3$ to get

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

and finally use $R_1 := R_1 - R_2$ to get the reduced row echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right].$$

(c) The leading variables are x_1, x_2 and x_4 whereas x_3 is a free variable. Taking $x_3 = t$ we find the general solution

$$x_1 = 0, \qquad x_2 = -2t, \qquad x_3 = t, \qquad x_4 = 2,$$

where t is an arbitrary scalar.

(d) Substituting into the new system we get

$$\begin{cases} -2\,b\,t + c\,t + 2\,d = 0 \\ -2\,a\,t + c\,t = 0 \\ d\,t + 2\,a = 2 \end{cases}$$

which holds for any value of t. Hence $c = 2b$, $d = 0$, $c = 2a$ and $a = 1$. Thus, the values of the parameters are $a = 1$, $b = 1$, $c = 2$, $d = 0$.

4. (10 marks).

(a) Formulate the *definition* of an eigenvalue and an eigenvector of a square matrix A.

(b) Prove that a scalar λ is an eigenvalue of a square matrix if and only if λ is a root of its characteristic polynomial.

(c) Consider the matrix

$$A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 4 & 3 & 1 \end{bmatrix}.$$

Without calculating all eigenvalues, explain why $\lambda = 0$ is an eigenvalue of A.

(d) Determine all eigenvalues of the matrix A.

(e) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of a certain matrix B with the respective eigenvalues λ and μ such that $\lambda \neq \mu$. Is it possible that $\mathbf{u} + \mathbf{v}$ is also an eigenvector of B? Justify your answer.

Solution

(a) A scalar λ is an eigenvalue of a matrix A if $A\mathbf{v} = \lambda\mathbf{v}$ for a certain nonzero vector $\mathbf{v}$. A nonzero vector $\mathbf{v}$ is an eigenvector of A if $A\mathbf{v} = \lambda\mathbf{v}$ for a certain scalar λ.

(b) If λ is an eigenvalue of a matrix A then $A\mathbf{v} = \lambda\mathbf{v}$ for a certain nonzero vector $\mathbf{v}$. Hence the homogeneous system of equations $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ has a nonzero solution $\mathbf{x} = \mathbf{v}$. Therefore, $\det(A - \lambda I) = 0$, that is, λ is a root of the characteristic polynomial. Conversely, if a scalar λ satisfies $\det(A - \lambda I) = 0$ then the homogeneous system of equations $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ has a nonzero solution, say, $\mathbf{x} = \mathbf{v}$. Then $\mathbf{v}$ is an eigenvector of A and λ is its eigenvalue.

(c) Since A has a row of zeros, we have $\det A = 0$. Therefore, $\lambda = 0$ is root of the characteristic polynomial.

(d) The characteristic polynomial of the matrix is

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 2 & 0 & 0 \\ 2 & 2-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ -1 & 4 & 3 & 1-\lambda \end{bmatrix}.$$

Expanding along the fourth column we get

$$\det(A - \lambda I) = (1-\lambda)\det \begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 2-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}.$$

Now expanding along the third column we get

$$\det(A - \lambda I) = (1-\lambda)\,(-\lambda)((2-\lambda)^2 - 4) = (4\lambda^2 - \lambda^3)(1-\lambda).$$

Hence, the eigenvalues are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$ and $\lambda_4 = 4$.

(e) This is impossible. Indeed, suppose on the contrary that $B(\mathbf{u}+\mathbf{v}) = \nu(\mathbf{u}+\mathbf{v})$ for a certain scalar ν. We have $B(\mathbf{u}+\mathbf{v}) = \lambda\mathbf{u}+\mu\mathbf{v}$ and so $\lambda\mathbf{u}+\mu\mathbf{v} = \nu(\mathbf{u}+\mathbf{v})$. This gives $(\lambda-\nu)\mathbf{u} = (\nu-\mu)\mathbf{v}$. However, $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of B with different eigenvalues and so they cannot be collinear. Hence $\lambda-\nu = 0$ and $\nu-\mu = 0$. This gives $\lambda = \mu = \nu$, a contradiction.

5. (10 marks).

(a) Find a lower triangular $n \times n$ matrix A with non-negative entries satisfying the condition $A\,A^T = C$, where C is the $n \times n$ matrix given by

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ & \cdots & & \cdots & & \cdots & & \cdots & \\ & \cdots & & \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}.$$

(b) Hence or otherwise calculate the determinant of C.

(c) Find the eigenvalues and corresponding eigenspaces of the matrix A found in part (a).

Solution

(a) Let a_{ij} denote the (i,j) entry of A. Then $a_{ij} = 0$ for all $i < j$. The condition $A\,A^T = C$ implies $a_{11}a_{i1} = 1$ for $i = 1, 2$ and $a_{11}a_{i1} = 0$ for $i \geq 3$. Since $a_{11} \geq 0$, we have $a_{11} = 1$ and so $a_{21} = 1$ while $a_{31} = \cdots = a_{n1} = 0$. Similarly, equating the entries in the second rows of $A\,A^T$ and C, we find that $a_{22} = a_{32} = 1$ while $a_{42} = \cdots = a_{n2} = 0$. Proceeding by induction we find

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & \cdots & & \cdots & & \cdots & & \cdots & \\ & \cdots & & \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

(b) We have $\det C = \det A \cdot \det A^T = (\det A)^2$. Since A is lower triangular, $\det A$ is the product of the diagonal entries, and so $\det C = 1$.
Alternatively, $\det C$ can be found by applying elementary row operations.

(c) All eigenvalues of A are equal to 1. The 1-eigenspace of A is spanned by the solutions of the equation $(A - I)\mathbf{x} = \mathbf{0}$. This system reads

$$x_2 = 0, \qquad x_3 = 0, \quad \ldots, \qquad x_n = 0.$$

Hence, the 1-eigenspace of A is

$$A = \begin{bmatrix} t \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad t \in \mathbb{R}.$$