Extended Answer Section

Answer these questions in the answer book(s) provided.
Ask for extra books if you need them.

1. (10 marks). Let π be the plane given by the equation $2x - 3y - 6z = 6$ and let M be the intersection point of the y axis and the plane π.

(a) Let π' be the plane through M with the normal vector $\mathbf{n} = 4\,\mathbf{j} + 3\,\mathbf{k}$. Find the Cartesian equations of the intersection line ℓ of the planes π and π'.

(b) Find the acute angle between the planes π and π'.

(c) Find parametric scalar equations of the line m which lies on the plane π, passes through the point M, and m is parallel to the xz coordinate plane.

(d) Find the coordinates of a point K on the line m such that the distance from K to the plane π' equals 3.

Solution

(a) Put $x = z = 0$ into the equation. This gives $M = (0, -2, 0)$. The line ℓ passes through M with the direction vector

$$(2\,\mathbf{i} - 3\,\mathbf{j} - 6\,\mathbf{k}) \times (4\,\mathbf{j} + 3\,\mathbf{k}) = 15\,\mathbf{i} - 6\,\mathbf{j} + 8\,\mathbf{k}.$$

Hence, the equations are

$$\frac{x}{15} = \frac{y+2}{-6} = \frac{z}{8}.$$

(b) For the cosine of the angle θ between the normal vectors we have

$$\cos\theta = \frac{(2\,\mathbf{i} - 3\,\mathbf{j} - 6\,\mathbf{k}) \cdot (4\,\mathbf{j} + 3\,\mathbf{k})}{|2\,\mathbf{i} - 3\,\mathbf{j} - 6\,\mathbf{k}|\,|4\,\mathbf{j} + 3\,\mathbf{k}|} = -\frac{30}{35} = -\frac{6}{7}.$$

Hence, the acute angle is $\cos^{-1}\dfrac{6}{7}$.

(c) The line m is the intersection line of the plane π and the plane $y = -2$. It passes through M with the direction vector

$$(2\,\mathbf{i} - 3\,\mathbf{j} - 6\,\mathbf{k}) \times \mathbf{j} = 6\,\mathbf{i} + 2\,\mathbf{k}.$$

Hence, the equations are

$$x = 3\,t, \qquad y = -2, \qquad z = t, \qquad\qquad t \in \mathbb{R}.$$

(d) Let $K = (3\,t, -2, t)$ for some $t \in \mathbb{R}$. The distance to the plane π' is the absolute value of the component $\overrightarrow{KM} \cdot \mathbf{v}/|\mathbf{v}|$, where $\mathbf{v} = 4\,\mathbf{j} + 3\,\mathbf{k}$. We have $\overrightarrow{KM} = t\,(3\,\mathbf{i} + \mathbf{k})$ and

$$\overrightarrow{KM} \cdot \mathbf{v}/|\mathbf{v}| = \frac{3\,t}{5}.$$

Hence, $t = 5$ or $t = -5$. The coordinates of K are then either $K = (15, -2, 5)$ or $K = (-15, -2, -5)$.

2. (10 marks). A matrix A and a column-vector $\mathbf{v}$ are given by

$$A = \begin{bmatrix} 1 & 2 & -4 & 1 \\ -1 & -2 & 4 & 0 \\ -2 & -4 & 8 & -1 \\ 1 & 3 & -4 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 6 \\ -4 \\ -10 \\ 8 \end{bmatrix}.$$

(a) Use elementary row operations to find the general solution of the system $A\mathbf{x} = \mathbf{v}$, where $\mathbf{x}$ is the column-vector with coordinates x_1, x_2, x_3, x_4.

(b) By using the elementary row operations of the previous part or otherwise find a column-vector $\mathbf{u}$ such that the system $A\mathbf{x} = \mathbf{u}$ is inconsistent.

(c) Is it possible to find a column-vector $\mathbf{u}$ such that the system $A\mathbf{x} = \mathbf{u}$ has a unique solution? Justify your answer.

(d) Prove that if $\mathbf{u}$ and $\mathbf{w}$ are two column-vectors such that each of the systems $A\mathbf{x} = \mathbf{u}$ and $A\mathbf{x} = \mathbf{w}$ is consistent, then the system $A\mathbf{x} = \alpha\,\mathbf{u} + \beta\,\mathbf{w}$ is consistent for any scalars α and β.

Solution

(a) The augmented coefficient matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & -4 & 1 & 6 \\ -1 & -2 & 4 & 0 & -4 \\ -2 & -4 & 8 & -1 & -10 \\ 1 & 3 & -4 & 1 & 8 \end{array}\right].$$

Using Gaussian elimination and starting with the elementary row operations $R_2 := R_2 + R_1$, $R_3 := R_3 + 2R_1$, $R_4 := R_4 - R_1$, we get the matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & -4 & 1 & 6 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right].$$

Now apply $R_2 \longleftrightarrow R_4$ and $R_4 := R_4 - R_3$ to get

$$\left[\begin{array}{cccc|c} 1 & 2 & -4 & 1 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This matrix is in row echelon form. In order to bring it into the reduced row echelon form, omit the row of zeros and apply $R_1 := R_1 - R_3$ and $R_1 := R_1 - 2R_2$ to get the reduced row echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right].$$

The leading variables are x_1, x_2 and x_4 whereas x_3 is a free variable. Taking $x_3 = t$ we find the general solution

$$x_1 = 4t, \qquad x_2 = 2, \qquad x_3 = t, \qquad x_4 = 2,$$

where t is an arbitrary scalar.

(b) A possible way to find such a column-vector is to replace the last column in the above row echelon form with the column with the entries 0, 0, 0, 1 and apply the reverse elementary row operations. This leads to the required column vector

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(This is not a unique solution.)

(c) Such a column-vector is impossible to find. The solution of the first part shows that the determinant of the matrix A is zero. Hence, for any given $\mathbf{u}$, the system $A\,\mathbf{x} = \mathbf{u}$ is either inconsistent or it has infinitely many solutions.

(d) Let column-vectors $\mathbf{y}$ and $\mathbf{z}$ satisfy $A\,\mathbf{y} = \mathbf{u}$ and $A\,\mathbf{z} = \mathbf{w}$. Then, by the rules of matrix multiplication,

$$A(\alpha\,\mathbf{y} + \beta\,\mathbf{z}) = \alpha\,A\mathbf{y} + \beta\,A\mathbf{z} = \alpha\,\mathbf{u} + \beta\,\mathbf{w}.$$

Hence, $\alpha\,\mathbf{y} + \beta\,\mathbf{z}$ is a solution of the system $A\mathbf{x} = \alpha\,\mathbf{u} + \beta\,\mathbf{w}$, so that the system is consistent.

3. (10 marks). Let A and B be two square matrices of the same size.

(a) Given that $\mathbf{v}$ is an eigenvector of A with the eigenvalue λ and $\mathbf{v}$ is an eigenvector of B with the eigenvalue μ, prove that $\mathbf{v}$ is an eigenvector of each of the matrices $A + B$ and AB and find the corresponding eigenvalues.

(b) Under the assumptions of the previous part, prove that $\det(AB - BA) = 0$.

(c) Find a common eigenvector $\mathbf{v}$ of the matrices

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}.$$

(d) Show that the vector $\mathbf{v}$ you found in the previous part is an eigenvector of the matrix $A^{-3}B^2$ and calculate the corresponding eigenvalue.

Solution

(a) We have $A\mathbf{v} = \lambda\,\mathbf{v}$ and $B\mathbf{v} = \mu\,\mathbf{v}$. Hence, $(A+B)\mathbf{v} = A\mathbf{v}+B\mathbf{v} = \lambda\,\mathbf{v}+\mu\,\mathbf{v} = (\lambda+\mu)\mathbf{v}$ and $AB\mathbf{v} = A(\mu\mathbf{v}) = \mu A\mathbf{v} = \lambda\mu\mathbf{v}$. Therefore, $\mathbf{v}$ is an eigenvector of each of the matrices $A + B$ and AB with the respective eigenvalues $\lambda + \mu$ and $\lambda\mu$.

(b) We have $(AB - BA)\mathbf{v} = AB\mathbf{v} - BA\mathbf{v} = \lambda\mu\mathbf{v} - \lambda\mu\mathbf{v} = \mathbf{0}$. This means the matrix $AB - BA$ has zero eigenvalue. Therefore, its determinant is zero.

(c) We have $\det(A - \lambda I) = \lambda^2 - 7\lambda + 6$, so the roots are $\lambda_1 = 6$ and $\lambda_2 = 1$. The corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Now, $\det(B-\lambda I) = \lambda^2+\lambda$, so the roots are $\lambda_1 = -1$ and $\lambda_2 = 0$. The corresponding eigenvectors are $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Hence, the common eigenvector of A and B is $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(d) Since $A\mathbf{v} = 6\mathbf{v}$ and A is invertible, we have $A^{-1}\mathbf{v} = \frac{1}{6}\mathbf{v}$. Hence, $A^{-3}B^2\mathbf{v} = (1/6)^3(-1)^2\,\mathbf{v} = (1/216)\,\mathbf{v}$. This means that $\mathbf{v}$ is an eigenvector of $A^{-3}B^2$ with the eigenvalue $1/216$.

4. (10 marks). Let X and Y be square matrices of the same size.

(a) Prove that the relation $(X+Y)^2 = X^2 + 2XY + Y^2$ implies $(X+Y)^3 = X^3 + 3X^2Y + 3XY^2 + Y^3$.

(b) Prove that if both matrices X and Y are invertible then the relation $(XY)^2 = X^2Y^2$ implies $(XY)^3 = X^3Y^3$.

(c) Would the statement in the previous part be true if only one of the matrices X or Y were invertible? Justify your answer.

Solution

See Assignment 2 solutions.

End of Extended Answer Section