THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Assignment 2

MATH1903: Integral Calculus and Modelling (Advanced) Semester 2, 2009

Lecturers: Holger Dullin and James Parkinson

This assignment is worth 5% of your overall assessment for MATH1903

The assignment is due on **Tuesday 6th October**

Please show all working, and present your arguments clearly. After all, mathematics is about communicating your ideas. This is a skill that takes time and effort to master.

Submission Instructions: Please put your assignment in the glass-fronted collection boxes on the verandah of Carslaw Level 3. These boxes are at the end of the verandah closest to Eastern Avenue – *not* the glass-fronted collection boxes near the pyramids on Carslaw Level 3, *nor* the open wooden pigeonholes.

Your assignment must be stapled inside a manilla folder, on the front of which you should write the initial of your family name as a LARGE letter. A cover sheet must be signed and attached. Please do not post your assignment before Tuesday 6th October, since the boxes are also used for the collection of assignments in other units.

1. [*2 marks*] Find a function $y(x)$ whose derivative is equal to its cosine squared and whose graph goes through the origin.

Solution: The differential equation for the unknown function $y(x)$ is $dy/dx = \cos^2 y$ and the initial condition is $y(0) = 0$. The equation can be solved by separation of variables, which gives $\int dy/\cos^2 y = \int dx$ and hence $\tan y = x + C$, therefore $y = \arctan(x + C)$. The initial condition gives $0 = \arctan C$, and hence $C = 0$. The final answer is $y = \arctan x$. Alternatively we can use definite integrals

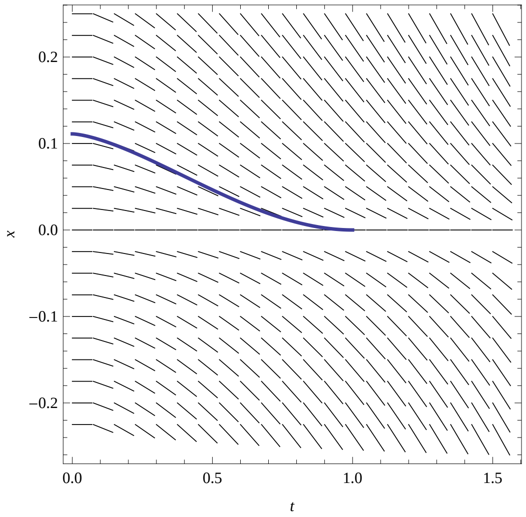
$$\int_0^y \frac{1}{\cos^2 r} dr = \int_0^x ds$$

which gives $\tan y - \tan 0 = x - 0$, and hence the same final answer.

2. [*7 marks*] Consider the differential equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -\sqrt{|tx|}\,.$

(a) [*2 marks*] Sketch the direction field for $t \in [0, 3/2]$ and $x \in [-1/4, 1/4]$.

Solution: (including the particular solution from part (b))



(b) [*2 marks*] Solve the differential equation to find the particular solution with $x(0) = 1/9$. Sketch this solution on your direction field.

Solution: Since the initial time is 0, we only need to consider non-negative times. Since the initial x is $1/9$ which is positive, we consider positive x, hence the absolute value sign can be omitted. Then separation of variables gives

$$\int \frac{\mathrm{d}x}{\sqrt{x}} = -\sqrt{t}\mathrm{d}t\,.$$

and integration gives $2\sqrt{x} = -2/3t^{3/2} + C$. The initial conditions gives $2\sqrt{1/9} = 0 + C$, and hence

$$x = \frac{1}{9}(1 - t^{3/2})^2\,.$$

The alternative computation with definite integrals is

$$\int_{1/9}^{x} \frac{\mathrm{d}r}{\sqrt{r}} = -\int_0^t \sqrt{s}\mathrm{d}s\,.$$

Since the solution is non-unique for $t > 1$ we only show the graph for $t \in [0, 1]$. A possible continuation would be the equilibrium solution. Simply plotting the solution formula for $t > 1$ gives an incorrect solution, as is obvious from the direction field; see part (d).

(c) [*1 mark*] Compute the time t it takes this solution to reach the equilibrium.

Solution: The equilibrium solution of the equation is $x = 0$. For the particular solution above this is reached when $t = 1$. In particular this time is finite.

(d) [*1 mark*] Find two solutions that start with the initial condition $x(1) = 0$.

Solution: One solution with $x(1) = 0$ is the equilibrium solution $x = 0$. The other solution is obtained from integrating (note that by inspecting the direction field we know that the solution can only leave towards negative values of x, so that now $|x|$ becomes $-x$ under the square root)

$$\int_0^x \frac{\mathrm{d}r}{\sqrt{-r}} = -\int_1^t \sqrt{s}\mathrm{d}s\,.$$

which gives

$$x = -\frac{1}{9}(1 - t^{3/2})^2 .$$

Up to the overall sign this is the same solution as in part (a).

(e) [*1 mark*] Explain why the uniqueness theorem does not apply.

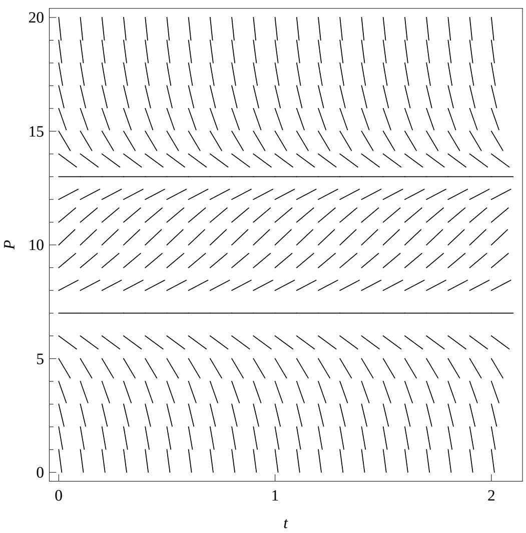
Solution: There are a number of ways to argue:
1) The right hand side of the equation does not have a continuous x-derivative at $x = 0$.
2) The function $1/\sqrt{|x|}$ does *not* satisfy the required bound $1/\sqrt{|x|} \leq K|x|$ for x near 0.
3) The main idea in the proof of the uniqueness theorem is to show that the time to reach (or leave) the equilibrium is not finite. However, in our case according to (b) the time is finite.

3. [*11 marks*] Modelling the number of fish $P(t)$ in a lake can be based on the logistic equation with an additional constant term $-c$ that describes the number of fish caught per unit time

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(20 - P) - c .$$

(a) [*1 mark*] Sketch the direction field in the positive quadrant for $c = 91$.

Solution: A clear direction field is obtained e.g. for $t \in [0, 2]$ and $P \in [0, 20]$:



(b) [*2 marks*] For general $c > 0$ find the equilibrium solutions. Depending on c there are either 0, 1, or 2 equilibria. For each case find the stability of the equilibria by considering the sign of $\mathrm{d}P/\mathrm{d}t$ nearby.

Solution: Equating the right hand side of the DE to zero gives the equilibrium solutions as roots of $P(20 - P) - c = 0$. This is a quadratic equation for P, and the solutions are $P_\pm = 10 \pm \sqrt{100 - c}$. The solutions $P(t) = P_-$ and $P(t) = P_+$ are equilibrium solutions of the DE when they are real, i.e. when $c \leq 100$. When $c = 100$ there is one equilibrium solution $P(t) = 10$. When $c > 100$ there are no equilibrium solutions.

Stability: For $c > 100$ there are no equilibria. For $0 \leq c < 100$ there are two equilibria P_- and P_+, where $0 < P_- < 10 < P_+$. Consider P near P_-. For

$P < P_-$ we have $\dot{P} < 0$ so $P(t) = P_-$ is unstable. Consider P near P_+. For $P < P_+$ we have $\dot{P} > 0$ and for $P > P_+$ we have $\dot{P} < 0$, so $P(t) = P_+$ is stable. For $c = 100$ there is one equilibrium at $P = 10$. For $P < 10$ we have $\dot{P} < 0$, so the equilibrium $P(t) = 10$ is unstable.

Remark: A change in the number and/or stability of equilibrium solutions in a DE is called a bifurcation. It represents a qualitative change in the behaviour of the solutions of a DE.

(c) [*8 marks*] Find the solution $P(t)$ with $P(0) = P_0 > 0$ in the two cases (i) $c = 75$ and (ii) $c > 100$. In each case either find the time τ after which the population is extinct, $P(\tau) = 0$, or find the limit of $P(t)$ as $t \to \infty$ (consider all $P_0 > 0$).

Solution:

Separating the variables gives

$$\int \frac{\mathrm{d}P}{P(20-P)-c} = \int \mathrm{d}t\,.$$

The integral on the left hand side is done by partial fractions. Depending on the value of c the quadratic polynomial in P in the denominator has either two real roots, a single double root, or complex roots, as analysed in the discussion of the equilibrium solutions.

(i) For $c = 75$ there are two real roots and partial fraction decomposition gives

$$\frac{1}{-(P-5)(P-15)} = \frac{1}{10(P-5)} - \frac{1}{10(P-15)}\,.$$

Integration from P_0 to P gives [1]

$$\frac{1}{10}\log\left|\frac{(P-5)}{(P_0-5)}\frac{(P_0-15)}{(P-15)}\right| = t\,.$$

From the qualitative analysis we know that if $P_0 < 5$ then $P < 5$, if $P_0 > 15$ then $P > 15$ and if $5 < P_0 < 15$ then $5 < P < 15$. Hence the absolute value sign can be omitted. Solving for P gives

$$P(t) = \frac{15(P_0-5)e^{10t} - 5(P_0-15)}{(P_0-5)e^{10t} - (P_0-15)} = 15 + \frac{10(P_0-15)}{(P_0-5)e^{10t} - (P_0-15)}\,.$$

When $P_0 > 5$ the solution $P(t)$ tends to the stable equilibrium $P = 15$ for $t \to \infty$. When $P_0 < 5$ the solution $P(t)$ is decreasing without bound, it reaches $P(\tau) = 0$ for $\tau = \frac{1}{10}\log\left(\frac{1}{3}(15-P_0)/(5-P_0)\right)$. Simply taking the limit $t \to \infty$ gives 15, but this answer is physically incorrect, since P went below 0 and through the pole at $t = \tau$ to get there.

(ii) For $c > 100$ the quadratic has complex roots. Completing the square gives $P(20-P) - c = -(P-10)^2 - (c-100)$. Since $c > 100$ we can set $c - 100 = d^2$. Then

$$t + C = \int \frac{-\mathrm{d}P}{(P-10)^2 + d^2} = -\frac{1}{d}\tan^{-1}\frac{P-10}{d}$$

[1] or use an indefinite integral, which requires some fiddling with the absolute values in the log's

and solving for P gives

$$P(t) = 10 - d\tan((t + C)d)\,.$$

The initial conditions implies $P_0 = 10 - d\tan(Cd)$ so that $Cd = -\tan^{-1}\frac{P_0 - 10}{d}$. The particular solution finally is

$$P(t) = 10 - d\tan\left(td - \tan^{-1}\frac{P_0 - 10}{d}\right), \quad d = \sqrt{c - 100}\,.$$

The solution is decreasing, and reaches extinction after some finite time τ. The time τ can be found by solving the equation $P(\tau) = 0$, or more directly by using definite integrals with the appropriate boundaries:

$$\int_{P_0}^{0} \frac{-\mathrm{d}P}{(P - 10)^2 + d^2} = \int_0^{\tau} \mathrm{d}t\,.$$

In either case, the result is

$$\tau = \frac{1}{d}\left(\tan^{-1}\frac{10}{d} + \tan^{-1}\frac{P_0 - 10}{d}\right)\,.$$

Notice that the use of the arctan addition formula is not valid here for all values of P_0 and $d > 0$. In particular also τd may be larger than $\pi/2$.