

Solutions to Assignment 1

MATH1903: Integral Calculus and Modelling (Advanced)

Semester 2, 2009

Lecturers: Holger Dullin and James Parkinson

This assignment is worth 5% of your overall assessment for MATH1903

The assignment is due on **Tuesday 18th August**

Please show all working, and present your arguments clearly. After all, mathematics is about communicating your ideas. This is a skill that takes time and effort to master.

Submission Instructions: Please put your assignment in the glass-fronted collection boxes on the verandah of Carslaw Level 3. These boxes are at the end of the verandah closest to Eastern Avenue – *not* the glass-fronted collection boxes near the pyramids on Carslaw Level 3, *nor* the open wooden pigeonholes.

Your assignment must be stapled inside a manilla folder, on the front of which you should write the initial of your family name as a LARGE letter. A cover sheet must be signed and attached. Please do not post your assignment before Tuesday 18th August, since the boxes are also used for the collection of assignments in other units.

1. Suppose we define the inverse tangent function by the integral (10 marks)

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2} dt \quad \text{for all } x \in \mathbb{R}.$$

Prove the following statements **directly from the above definition**:

- (a) If $x \in \mathbb{R}$ then $\tan^{-1}(-x) = -\tan^{-1}(x)$.

Solution: From the definition we have

$$\tan^{-1}(-x) = \int_0^{-x} \frac{1}{1+t^2} dt = \int_0^x \frac{1}{1+(-u)^2} (-1) du = -\tan^{-1}(x),$$

where we have made a change of variable $t = -u$ in the integral

- (b) If $a, b \geq 0$ and $ab < 1$ then

$$\tan^{-1}(a) + \tan^{-1}(b) = \tan^{-1}\left(\frac{a+b}{1-ab}\right).$$

Hint: Use The Fundamental Theorem of Calculus to differentiate $\tan^{-1}\left(\frac{a+x}{1-ax}\right)$.

Solution: Using the chain rule and the Fundamental Theorem of Calculus we have

$$\frac{d}{dt} \tan^{-1} f(x) = f'(x) \frac{1}{1+f(x)^2}.$$

Applying this to $f(x) = \frac{a+x}{1-ax}$ (as suggested by the hint) gives

$$\begin{aligned} \frac{d}{dx} \tan^{-1} \left(\frac{a+x}{1-ax} \right) &= \frac{d}{dx} \left(\frac{a+x}{1-ax} \right) \frac{1}{1 + \left(\frac{a+x}{1-ax} \right)^2} \\ &= \frac{1+a^2}{(1-ax)^2} \frac{1}{1 + \left(\frac{a+x}{1-ax} \right)^2} \\ &= \frac{1+a^2}{(1-ax)^2 + (a+x)^2} \\ &= \frac{1}{1+x^2}. \end{aligned}$$

Integrating both sides over the interval $[0, b]$ gives

$$\int_0^b \frac{d}{dx} \tan^{-1} \left(\frac{a+x}{1-ax} \right) dx = \int_0^b \frac{1}{1+x^2} dx.$$

Using the Fundamental Theorem of Calculus for the integral on the left hand side, and the definition of \tan^{-1} for the integral on the right hand side, we get

$$\tan^{-1} \left(\frac{a+x}{1-ax} \right) \Big|_0^b = \tan^{-1}(x) \Big|_0^b,$$

and so

$$\tan^{-1} \left(\frac{a+b}{1-ab} \right) - \tan^{-1}(a) = \tan^{-1}(b) - \tan^{-1}(0) = \tan^{-1}(b),$$

which rearranges to give the result.

Remark: Note that the condition $ab < 0$ is used to ensure that $\tan^{-1} \left(\frac{a+x}{1-ax} \right)$ is defined on the integration range $[0, b]$.

2. Let $a < b$ and let f be a bounded function on $[a, b]$. **(10 marks)**

(a) Let P and Q be partitions of $[a, b]$. Then P is a *refinement* of Q if $Q \subseteq P$. Show that if P is a refinement of Q then

$$L_P \geq L_Q \quad \text{and} \quad U_P \leq U_Q,$$

where U_P is the upper Riemann sum of f using the partition P , and L_P is the lower Riemann sum of f using the partition P (similarly for U_Q and L_Q).

Solution: Suppose that $P = \{p_0, \dots, p_m\}$ is a refinement of $Q = \{q_0, \dots, q_n\}$. for $j = 1, \dots, m$ and $k = 1, \dots, n$, define

$$\begin{aligned} m_j^{(P)} &= \min\{f(x) \mid p_{j-1} \leq x \leq p_j\}, & M_j^{(P)} &= \max\{f(x) \mid p_{j-1} \leq x \leq p_j\}, \\ m_k^{(Q)} &= \min\{f(x) \mid q_{k-1} \leq x \leq q_k\}, & M_k^{(Q)} &= \max\{f(x) \mid q_{k-1} \leq x \leq q_k\}. \end{aligned}$$

Therefore the upper and lower Riemann sums for the partitions P and Q are

$$\begin{aligned} L_P &= \sum_{j=1}^m m_j^{(P)} \Delta p_j, & U_P &= \sum_{j=1}^m M_j^{(P)} \Delta p_j, \\ L_Q &= \sum_{k=1}^n m_k^{(Q)} \Delta q_j, & U_Q &= \sum_{k=1}^n M_k^{(Q)} \Delta q_j. \end{aligned}$$

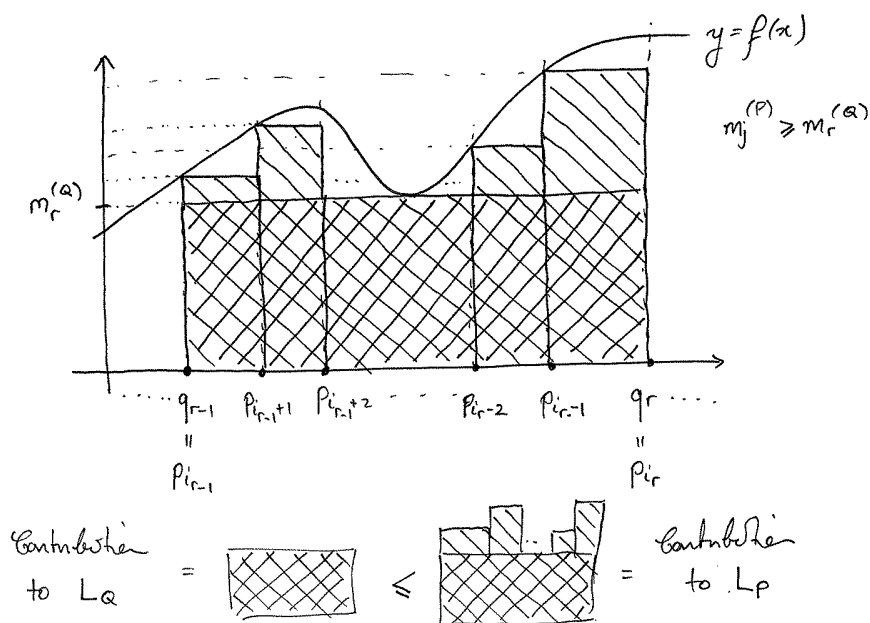
Since $Q \subseteq P$ there is a sequence

$$0 = i_0 < i_1 < i_2 < \dots < i_n = m$$

of indices such that

$$\begin{aligned} [p_{i_0}, p_{i_1}] &= [q_0, q_1] \\ [p_{i_1}, p_{i_2}] &= [q_1, q_2] \\ [p_{i_2}, p_{i_3}] &= [q_2, q_3] \\ &\vdots \\ [p_{i_{n-1}}, p_{i_n}] &= [q_{n-1}, q_n]. \end{aligned}$$

That is, $[p_{i_{r-1}}, p_{i_r}] = [q_{r-1}, q_r]$ for all $r = 1, \dots, n$. Let $r = 1, \dots, n$. Let us consider the contributions to the lower Riemann sums L_Q and L_P coming from the interval $[q_{r-1}, q_r]$. The following picture should help to make the argument more transparent.



The contribution to the Riemann sum L_Q from the interval $[q_{r-1}, q_r]$ is

$$m_r^{(Q)} \Delta q_r.$$

The contribution to the Riemann sum L_P from the interval $[q_{r-1}, q_r]$ is the sum of the contributions from the smaller intervals:

$$[q_{r-1}, q_r] = [p_{i_{r-1}}, p_{i_r}] = [p_{i_{r-1}}, p_{i_{r-1}+1}] \cup [p_{i_{r-1}+1}, p_{i_{r-1}+2}] \cup \dots \cup [p_{i_r-1}, p_{i_r}].$$

Hence the contribution to L_P from the interval $[q_{r-1}, q_r]$ is

$$\sum_{j=i_{r-1}+1}^{i_r} m_j^{(P)} \Delta p_j.$$

Since $[p_{j-1}, p_j] \subseteq [q_{r-1}, q_r]$ for all $j = i_{r-1} + 1, \dots, i_r$ we have $m_j^{(P)} \geq m_r^{(Q)}$ for all $j = i_{r-1} + 1, \dots, i_r$, and hence

$$\sum_{j=i_{r-1}+1}^{i_r} m_j^{(P)} \Delta p_j \geq m_r^{(Q)} \sum_{j=i_{r-1}+1}^{i_r} (p_j - p_{j-1}) = m_r^{(Q)} \Delta q_r$$

(note that the sum collapses). This shows that the contribution to L_P from the interval $[q_{r-1}, q_r]$ is greater than or equal to the contribution to L_Q from the interval $[q_{r-1}, q_r]$. This was clear from the picture. Summing over all $r = 1, \dots, n$ gives $L_P \geq L_Q$.

The argument for the upper Riemann sums is similar: The contribution to U_P from the interval $[q_{r-1}, q_r]$ is

$$\sum_{j=i_{r-1}+1}^{i_r} M_j^{(P)} \Delta p_j \leq M_r^{(Q)} \sum_{j=i_{r-1}+1}^{i_r} \Delta p_j = M_r^{(Q)} \Delta q_r,$$

and since the right hand side is the contribution to U_Q from the interval $[q_{r-1}, q_r]$ we see that $U_P \leq U_Q$.

- (b) Let P_1 and P_2 be arbitrary partitions of $[a, b]$. Show that

$$L_{P_1} \leq U_{P_2}.$$

Hint: Let P be the partition $P = P_1 \cup P_2$. Then P is a refinement of both P_1 and P_2 (called the common refinement of P_1 and P_2). Now use (a).

Solution: Let $P = P_1 \cup P_2$. Therefore P is a refinement of both P_1 and P_2 . Therefore by part (a) we have

$$L_{P_1} \leq L_P \quad \text{and} \quad U_P \leq U_{P_2}.$$

Since $L_P \leq U_P$ (this is obviously true for any partition) we have

$$L_{P_1} \leq L_P \leq U_P \leq U_{P_2}$$

as required.